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EQUATIONS OF MOTION FOR HYDROFOIL CRAFT

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NOMENCLATURE

$\vec{a}_c = (a_{cx}, a_{cy}, a_{cz})$ Acceleration of boat C.G.

a_{ij} Influence coefficient of boat, the elastic deflection at point i due to unit force at point j.

$\bar{c}_c$ Mean chord measured normal to hinge line of that portion of control surface and the area that lies aft of hinge line.

e_c Normal distance from mass center of control surface to hinge line (see table of subscripts below).

$f_i^{(e)}$ z-component of displacement of i-th element due to elastic deformation.

$\ddot{f}_t$ z-component of perturbation acceleration from equilibrium flight position of an element of boat volume $d\tau$.

g Acceleration of gravity.

$g_i^{(e)}$ x-component of displacement of i-th element due to elastic deformation.

$\ddot{g}_t$ x-component of perturbation acceleration from equilibrium flight position of an element of boat volume $d\tau$.

h Depth of hydrofoil - positive down.

$\vec{h} = (h_x, h_y, h_z)$ Angular momentum vector of boat.

$\vec{h}_R = (h_{xR}, h_{yR}, h_{zR})$ Vector angular momentum of rotor relative to boat axes.

$\vec{i}, \vec{j}, \vec{k}$ Orthogonal unit vectors in direction of x,y,z axes respectively.

ℓ Reference length, usually the distance between two convenient points on the forward and aft hydrofoil systems. For the sake of uniformity it is recommended that the longitudinal distance between the forward-most points of the forward and aft foil systems be used; e.g. distance between leading edge of root chords.

ℓ_c Distance from boat C.G. to control stock (see Figure 2), (see table of subscripts below).

m Mass of boat.

$\underline{m}$ Diagonal mass distribution matrix.

m_c Total mass of control surface (see table of subscripts below).

$\vec{n} = (n_x, n_y, n_z)$ Unit vector normal outward to surface element.

p Pressure.

p Perturbation in P.

$\dot{p}$ Time rate of change of perturbation of P.

q Perturbation in Q.

$\dot{q}$ Time rate of change of q.

$\vec{r} = (x, y, z)$ Position vector of a point on the boat relative to its C.G.

r Perturbation in R.

$\dot{r}$ Time rate of change of r.

s Variable of Laplace transform (see Equation [58]).

t Time.

u Perturbation in U.

$\dot{u}$ Time rate of change of u.

v Perturbation in V.

$\dot{v}$ Time rate of change of v.

w Perturbation in W.

$\dot{w}$ Time rate of change of w.

x, y, z The boat stability axes. A right-hand orthogonal system of moving axes, fixed in the boat. The z-axis is directed toward the bottom of the boat, the xz plane lies in the vertical plane of symmetry of the boat, the origin o is located at the center of mass of the boat and the x-axis is forward and coincides with the direction of motion when the boat is in equilibrium flight.

x_a, y_a, z_a A system of axes similar to the x,y,z axes except that the axes lie along the principal axes of inertia through the mass center of the system.

x_o, y_o, z_o A right-hand orthogonal system of axes fixed relative to the equilibrium water surface. The $x_o y_o$ plane is fixed parallel to the equilibrium plane of the free water surface, the z_o direction is vertically down and the x_o axis lies in the general direction of the initial motion of the boat.

A Reference area, usually projected area of foil system on x-y plane of body axes at design condition.

A_c Area of that portion of control surface and tab area that lies aft of hinge line.

$\underline{A} = \underline{A_{ij}}$ Hydrodynamic added mass square matrix arising from acceleration motion in the elastic degrees of freedom.

$\underline{B} = \underline{B_{ij}}$ Hydrodynamic force square matrix arising from velocity motion in the elastic degrees of freedom.

$\underline{C} = \underline{C_{ij}}$ Hydrodynamic force square matrix arising from torsion in the elastic degrees of freedom.

$D = \frac{d}{dt}$ Differential time operator.

$D^2 = \frac{d^2}{dt^2}$ Differential time operator.

$D_u = \frac{\partial}{\partial u}$ Typical partial derivative operator with respect to u.

F_c Moment applied to control by all power sources (see table of subscripts below).

F_n Generalized force driving the n-th normal mode (see Equation [66]).

$\vec{F} = (F_x, F_y, F_z)$ Externally applied force vector.

$\vec{G} = (K, M, N)$ Externally applied moment vector.

G_{Zw} Typical hydrodynamic transfer function (see Equation [58]).

H_c Hydrodynamic moment about control stock (see table of subscripts below).

H_{cu} Typical rate of change of control hinge moment with u; the partial derivative $\frac{\partial H_c}{\partial u}$ (see table of subscripts below); (see note below).

$\underline{I}$ Unit diagonal matrix.

I_c Effective mome nt of inertia of the control system relative to control stock (see table of subscripts below).

I_n Generalized inertia in n-th mode (see Equation [67]).

I_x Moment of inertia of boat about its stability x-axis.

I_{xa} Moment of inertia of boat about its principal x_a-axis.

I_y Moment of inertia of boat about its stability y-axis.

I_z Moment of inertia of boat about its stability z-axis.

I_{za} Moment of inertia of boat about its principal z_a-axis.

I_{zx} Product of inertia of boat with respect to its zx-axes.

J_c Moment applied to control surface by manual power (see table of subscripts below); (see note below).

K Hydrodynamic moment component about the stability x-axis (positive from y to z) - rolling moment.

$K_1^{(e)}$ The contribution of the elastic degrees of freedom in the lateral modes to the total roll moment.

$K_2^{(e)}$ The contribution of the elastic degrees of freedom in the lateral antisymmetric modes to the total roll moment.

M Hydrodynamic moment component about the stability y-axis (positive from z to x) - pitching moment.

$M^{(e)}$ The contribution of the elastic degrees of freedom to the total M-moment.

N Hydrodynamic moment component about the stability z-axis (positive from x to y) - yawing moment.

$N_1^{(e)}$ The contribution of the elastic degrees of freedom in the lateral modes to the total yaw moment.

P Component of $\vec{\omega}$ about x-axis (see note below).

P_{ay} Product of inertia term for ailerons (see Equation [30]).

P_c Moment applied to control surface by power sources other than manual (see table of subscripts below).

P_{ex} Product of inertia of elevator, defined in Equation [24].

P_{rx} Moment of inertia term for rudder (see Equation [27].

P_{rz} Product of inertia term for rudder (see Equation [27].

Q Component of $\vec{\omega}$ about y-axis (see note below).

R Component of $\vec{\omega}$ about z-axis (see note below).

T Kinetic energy.

U Elastic strain energy.

U Component of $\vec{U}_o$ along x-axis.

$\vec{U}_o = (U,V,W)$ Velocity vector of boat center of gravity.

$\forall$	Volume
$\dot{V}$	Time rate of change of V
W	Weight of boat
$\dot{W}$	Component of U_o along z-axis.
X	Hydrodynamic force in the positive direction of the stability x-axis - longitudinal force.
X_H	Hydrodynamic component of force per unit volume in x-direction.
X_{Hi}	Hydrodynamic component of force on the i-th element in the x-direction.
X_δ	Rate of change of X hydrodynamic force with control deflection δ; the partial derivative $\frac{\partial X}{\partial \delta}$.
Y	Hydrodynamic force in the positive direction of the stability y-axis - lateral force.
$Y_1^{(e)}$	The contribution of the elastic degrees of freedom in the lateral direction to the total Y-force.
Z	Hydrodynamic force in the positive direction of the stability z-axis - vertical force.
Z_H	Hydrodynamic component of force per unit volume in z-direction.
Z_{Hi}	Hydrodynamic component of force on the i-th element in the z-direction.
Z_i	Net Z-force at element i.
Z_j	Net Z-force at element j.
Z_R	The contribution of the rigid body motions to the Z-force.
$Z^{(e)}$	The contribution of the elastic degrees of freedom to the total Z-force.

δ_c	Control surface rotation angle about hinge line (see Figure 2 and note below); (see table of subscripts below).
ϵ	Angle between the principal x_a-axis and the stability x-axis positive when the x_a-axis is pointed above the x-axis forward of the C.G.
$\epsilon_n(t)$	Generalized coordinate giving displacement of n-th normal mode (function of time t).
ζ	Perturbation in rudder angle (see note below).
$\dot{\zeta}$	Time rate of change of ζ.
η	Perturbation in elevator angle (see note below).
$\dot{\eta}$	Time rate of change of η.
Θ	Pitch angle (see Section I-3 and Figure 1).
Λ_h	Sweep angle of control surface hinge line.
ξ	Perturbation in aileron angle (see note below).
$\dot{\xi}$	Time rate of change of ξ.
ρ	Mass density of water, slugs per cu. ft.
ρ_B	Average mass density of an element of boat volume $d\tau$.
ρ_c	Normal distance to mass element from hinge line of control surface (see Figure 2), (see table of subscripts below).
τ	Volume of boat; $d\tau$ volume element.
Φ	Roll angle (see Section I-3 and Figure 1).
Ψ	Angle of heading (see Section I-3 and Figure 1).
$\vec{\omega}$	(P,Q,R) = Angular velocity vector of boat (see note below).
ω_n	Natural frequency of n-th mode (Rad./sec.).
θ	Perturbation in Θ
ϕ	Perturbation in Φ

MATRIX NOTATION

$\underline{1}'$	=	Unit row matrix which operates so as to sum all the terms of the column matrix on its right.
$\underset{\_}{z}'$	=	Row matrix of z.
{ }	=	Column matrix.
[ ]	=	Square matrix.

STATIC DERIVATIVES

X_u, X_v, X_w, X_h — Rate of change of X hydrodynamic force with u, v, w or h at steady state equilibrium condition; the partial derivative $\frac{\partial X}{\partial u}$, $\frac{\partial X}{\partial v}$, $\frac{\partial X}{\partial w}$ or $\frac{\partial X}{\partial h}$.

Y_u, Y_v, Y_w, Y_h

Z_u, Z_v, Z_w, Z_h

K_u, K_v, K_w, K_h

M_u, M_v, M_w, M_h

N_u, N_v, N_w, N_h

ROTARY DERIVATIVES

$X_p, X_q, X_r, X_\theta, X_\phi$ — Rate of change of X hydrodynamic force with p, q, r, θ or ϕ at steady state equilibrium condition; the partial derivative; $\frac{\partial X}{\partial p}$, $\frac{\partial X}{\partial q}$, $\frac{\partial X}{\partial r}$, $\frac{\partial X}{\partial \theta}$ or $\frac{\partial X}{\partial \phi}$.

$Y_p, Y_q, Y_r, Y_\theta, Y_\phi$

$Z_p, Z_q, Z_r, Z_\theta, Z_\phi$

$K_p, K_q, K_r, K_\theta, K_\phi$

$M_p, M_q, M_r, M_\theta, M_\phi$

$N_p, N_q, N_r, N_\theta, N_\phi$

ADDED MASS DERIVATIVES

$X_{\dot{u}}, X_{\dot{v}}, X_{\dot{w}}$	Rate of change of X hydrodynamic force with $\dot{u}$, $\dot{v}$, or $\dot{w}$; $\frac{\partial X}{\partial \dot{u}}$, $\frac{\partial X}{\partial \dot{v}}$, or $\frac{\partial X}{\partial \dot{w}}$.
$Y_{\dot{u}}, Y_{\dot{v}}, Y_{\dot{w}}$	
$Z_{\dot{u}}, Z_{\dot{v}}, Z_{\dot{w}}$	
$X_{\dot{p}}, X_{\dot{q}}, X_{\dot{r}}$	
$Y_{\dot{p}}, T_{\dot{q}}, Y_{\dot{r}}$	
$Z_{\dot{p}}, Z_{\dot{q}}, Z_{\dot{r}}$	

ADDED MOMENT OF INERTIA DERIVATIVES

$K_{\dot{u}}, K_{\dot{v}}, K_{\dot{w}}$	Rate of change of K hydrodynamic moment with $\dot{u}$, $\dot{v}$ or $\dot{w}$; the partial derivative $\frac{\partial K}{\partial \dot{u}}$, $\frac{\partial K}{\partial \dot{v}}$ or $\frac{\partial K}{\partial \dot{w}}$.
$M_{\dot{u}}, M_{\dot{v}}, M_{\dot{w}}$	
$N_{\dot{u}}, N_{\dot{v}}, N_{\dot{w}}$	
$K_{\dot{p}}, K_{\dot{q}}, K_{\dot{r}}$	
$M_{\dot{p}}, M_{\dot{q}}, M_{\dot{r}}$	
$N_{\dot{p}}, N_{\dot{q}}, N_{\dot{r}}$	

SUBSCRIPTS

bar, —	= square matrix
i	= initial reference value
a	= aileron
e	= elevator
r	= rudder
n	= the n-th of a series

SUPERSCRIPTS

dot, ·	=	derivative with respect to time
prime, '	=	represents non-dimensional quantity (see Table I)
bar, —	=	Laplace transform; also mean value
arrow, →	=	vector

Note - All rotations (control surfaces, angular velocity components, etc.) and moment components are positive according to the usual right-handed convention; i.e., from the x-axis to the y-axis to the z-axis to the x-axis.

ABSTRACT

The hydrofoil boat in a seaway is a very complicated system. In its simplest form it may be treated as a rigid body. However, this representation is not always correct. In some cases it may be necessary to take into account other factors such as the gyroscopic effects of propellers, the dynamics of the control systems and the elastic behavior of the boat. In this report the general equations of motion of the hydrofoil boat, treated as a rigid body, are first derived and discussed. The gyroscopic effects of the propellers, the dynamics of the control systems and the effects of the elasticity of the boat are then considered and related to the equations of motion of the boat.

EQUATIONS OF MOTION OF A HYDROFOIL BOAT

INTRODUCTION

In order to study the effect of various parameters on the stability, control and motion in a seaway of hydrofoil craft, it is necessary to solve a set of mathematical equations which relate the physical properties of the hydrofoil boat with the hydrodynamic forces resulting from its own motion and that of the seaway. The fundamental equations used in such analyses appear in various sources and with varying degrees of generality often without derivation. It is the purpose of this report to present, in one place, a collection of the basic equations of motion of the hydrofoil boat, together with brief derivations; but in sufficient detail to afford a thorough understanding of these equations. No attempt is made, however, to present here methods of estimating the hydrodynamic forces.

In the first part of this report the general equations of motion of the boat, treated as a rigid body with spinning rotors, are derived in terms of a set of axes fixed in the boat. These are six non-linear scalar equations representing the components of dynamic equilibrium in each of the six degrees of freedom. Six additional equations relating the motions of the boat in body axes to the orientation and motion of the boat with respect to fixed axes are then derived. The above mentioned set of twelve non-linear equations characterize the rigid boat with spinning rotors and are sufficient to determine its response to an arbitrary set of time dependent forces and moments. In addition the equations characterizing the dynamic response of each of the major control systems are derived.

The second part of this report discusses the applications of the foregoing set of equations to the determination of the stability, control and response to a seaway of a hydrofoil boat. Large cross coupling terms, in these equations, between the various degrees of freedom suggest the need for solving all the equations of motion simultaneously on high speed digital computers when large motions such as those obtained in a heavy or sometimes even mild following seas are of interest. Where only longitudinal motions are of interest considerable simplification may be achieved. Further for the class of problems in which the boat perturbations from equilibrium are small the equations of motion may be linearized and thus lead to even greater computational simplification with acceptable accuracy. Such linearization also makes the study of automatic controls and hydroelastic effects more tractable.

The third part of this report presents the linearized equations of motion of the rigid hydrofoil boat and their derivation. These equations are also derived in non-dimensional form and in terms of hydrodynamic transfer functions. The latter set of equations are of particular importance when studies of automatic controls and effects of unsteadiness on boat stability and motions are of interest.

The fourth part of this report discusses some of the methods which may be used for determining the effect of the elastic deformation of the boat on its stability and control. The best known of these are the method of normal modes, the collocation approach and the method of assumed modes. The linearized equations of motion of the boat including its motion in the elastic degrees of freedom are derived by the first two methods. A set of these three simultaneous linear equations are obtained corresponding to the

original three rigid body degrees of freedom in the longitudinal plane (or lateral plane) plus n degrees of freedom in the elastic modes. For the case where the natural frequencies of vibration of the structure are much higher than the frequencies of the rigid-body motions the hydrodynamic loading may be considered slow enough so that the structure is always in static equilibrium. This simplification leads to the quasisteady equations of motion. The steady flight equilibrium equations in which the hydroelastic effects are taken into account are then given as a special case of the former set of equations.

Much of the material has been adapted from standard references. In particular, Reference 14 has served as a valuable guide.

I. DERIVATION OF EQUATIONS OF MOTION OF RIGID BOAT

1. Rigid Body Equations

In this Section the hydrofoil boat is considered as a rigid body and a derivation of its equations of motion from first principles is presented.

From Newton's second law, the force $\vec{F}$ acting through the center of gravity c of the boat, in vector notation is

$$\vec{F} = m \frac{d\vec{U}_o}{dt} \qquad [1]$$

where m is the mass of the boat and $\vec{U}_o$ its velocity. The moment of the boat $\vec{G}$ is equal to the time rate of change of its angular momentum and is given by

$$\vec{G} = \frac{d\vec{h}}{dt} \qquad [2]$$

where the angular momentum $\vec{h}$ is, in terms of a particle of mass dm located a vector distance $\vec{r}$ from the center of gravity,

$$\vec{h} = \int_{\forall} [\ \vec{r} \times (\ \vec{U}_o + \vec{\omega} \times \vec{r}\ )\ ]\ dm$$

where the integration is taken over the whole boat. Since $\vec{U}$ is the same for all particles and $\int \vec{r}\, d\, m = 0$ the above equation becomes, after expanding the vector triple product,

$$\vec{h} = \int [\, \vec{\omega}\, (\vec{r}\cdot\vec{r}) - \vec{r}\, (\vec{\omega}\cdot\vec{r})\,]\, dm \quad .$$

Let $\vec{r}$ and $\vec{\omega}$ be expressed by

$$\begin{aligned} \vec{r} &= x\,\vec{i} + y\,\vec{j} + z\,\vec{k} \\ \vec{\omega} &= P\,\vec{i} + Q\,\vec{j} + R\,\vec{k} \end{aligned} \qquad [3]$$

where x, y and z are the scalar components of $\vec{r}$, P, Q and R are the scalar components of $\vec{\omega}$ and $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors in the direction of x, y and z. Substituting these expressions in Equation 3 gives for the scalar components of $\vec{h}$, in terms of the moments and products of inertia A, B, ... F,

$$\begin{aligned} h_x &= I_x\, P - I_{xy}\, Q - I_{zx}\, R \\ h_y &= -\, I_{xy}\, P + I_y\, Q - I_{yz}\, R \\ h_z &= -\, I_{zx}\, P - I_{yz}\, Q + I_z\, R \end{aligned} \qquad [4]$$

where

$$\begin{aligned} I_x &= \int (y^2 + z^2)\, dm, & I_{yz} &= \int yz\, dm, \\ I_y &= \int (z^2 + x^2)\, dm, & I_{zx} &= \int zx\, dm, \\ I_z &= \int (x^2 + y^2)\, dm, \quad \text{and} & I_{xy} &= \int xy\, dm\ . \end{aligned}$$

If x, y and z are taken as the coordinates of dm in a non-rotating frame of reference, with origin at the center of gravity of the boat, C, then it is clear that in general the moments and products of inertia as well as the angular velocity components P, Q and R will vary with time as the boat rotates.

This is an unnecessary complication which can be avoided if the coordinate system 0 x y z is fixed in the boat and allowed to rotate with it. Though this introduces additional terms the resulting equations are much simpler since now the inertia terms will remain constant. Thus Equation [1] becomes in terms of the velocity components of $\vec{U}_o$ (U,V,W) and angular velocity $\vec{\omega}$

$$\vec{F} = m \frac{d}{dt} (U \vec{i} + V \vec{j} + W \vec{k})$$

$$= m \left[(\dot{U} \vec{i} + \dot{V} \vec{j} + \dot{W} \vec{k}) + (U \frac{d \vec{i}}{d t} + V \frac{d \vec{j}}{d t} + W \frac{d \vec{k}}{d t}) \right]$$

$$= m \left[\frac{\delta \vec{U}_o}{\delta t} + \vec{\omega} \times \vec{U}_o \right] \qquad [5]$$

since $\frac{d \vec{i}}{d t} = \vec{\omega} \times \vec{i}$, $\frac{\partial \vec{j}}{\partial t} = \vec{\omega} \times \vec{j}$ and $\frac{d \vec{k}}{d t} = \vec{\omega} \times \vec{k}$. The operator $\frac{\delta}{\delta t}$ has the definition implied by Equation [5]. Similarly, Equation [2] becomes

$$\vec{G} = \frac{\delta \vec{h}}{\delta t} + \vec{\omega} \times \vec{h} . \qquad [6]$$

Combining Equations [5] and [6] with Equation [3] gives for the scalar components of $\vec{F}$ in the $\vec{i}$, $\vec{j}$ and $\vec{k}$ directions

$$F_x = m [\dot{U} + Q W - R V]$$

$$F_y = m [\dot{V} + R U - P W]$$

$$F_z = m [\dot{W} + P V - Q U] \qquad [7]$$

and for the scalar components of $\vec{G}$ in the $\vec{i}$, $\vec{j}$ and $\vec{k}$ directions

$$K = \dot{h}_x + Q\,h_z - R\,h_y$$

$$M = \dot{h}_y + R\,h_x - P\,h_z$$

$$N = \dot{h}_z + P\,h_y - Q\,h_x \qquad [8]$$

Equations [7] and [8] are the Euler equations of motion of the hydrofoil boat.

2. Effect of Spinning Rotors

In the derivation of the Euler equations the boat was assumed to be a rigid body. When there are spinning rotors attached to the boat such as propellers and engine rotors, their effect on the boat motion may sometimes be significant enough to require their inclusion in the equations of motion. In this case it becomes necessary to modify Equation [4] to include the effect of the angular momentum of the rotors. It can be shown that the total angular momentum of the boat is simply the vector sum of that given for the boat treated as a rigid body and the angular momentum of each rotor relative to the body axis. The latter is determined from Equation [4] by interpreting the moments and products of inertia therein as those of the rotor with respect to axes parallel to O x y z and origin at the center of gravity of the rotor. Thus if the resultant of the relative angular momenta of all rotors is $\vec{h}_R$ with components h_{xR}, h_{yR}, h_{zR} then the total angular momentum (Equation [4]) becomes

$$h_x = I_x P - I_{xy} Q - I_{zx} R + h_{xR}$$

$$h_y = - I_{xy} P + I_y Q - I_{yz} R + h_{yR}$$

$$h_z = - I_{zx} P - I_{yz} Q + I_z R + h_{zR} \qquad [9]$$

and the moment equations (Equation [8]) become

$$K = \dot{h}_x + Q h_z - R h_y + Q h_{zR} - P h_{yR}$$

$$M = \dot{h}_y + R h_x - P h_z + R h_{xR} - P h_{zR}$$

$$N = \dot{h}_z + P h_y - Q h_x + P h_{yR} - Q h_{xR} \qquad [10]$$

where the additional terms are known as gyroscopic couples. For example suppose the axis of a rotor, having a moment of inertia I and angular velocity Ω, is in the direction of body axis O_x, then $\vec{h}_R = \vec{i}\, I\, \Omega$ and the components of the gyroscopic couple are 0, $I\,\Omega\,R$ and $- I\,\Omega\,Q$.

3. Motion of Boat Relative to Fixed Coordinate

Solution of Equations [7] and [8] gives the linear velocity components U, V, W and angular velocity components P, Q, R relative to the O x y z axes fixed in the boat. To obtain the motion of the boat center of gravity it is necessary to express the linear velocities relative to a fixed coordinate system.

We therefore define a fixed orthogonal coordinate system $O\, x_o\, y_o\, z_o$ in which the $x_o\, y_o$ plane is fixed parallel to the equilibrium plane of the free water surface and the z_o direction

is positive downward. The orientation of the boat axes (x,y,z) relative to the fixed axes (x_o, y_o, z_o) is shown in Figure 1. It is assumed that at first the two reference frames are parallel. Then the orientation of the boat is determined by considering the following three rotations, in the order indicated, where all rotations are in the positive direction.

i. Rotate the x_o, y_o axes about z_o through the angle of yaw Ψ to the position (x_1, y_1). Then the direction cosines between (x_o, y_o, z_o) and (x_1, y_1, z_1) are given by

	x_1	y_1	z_1
x_o	$\cos \Psi$	$-\sin \Psi$	0
y_o	$\sin \Psi$	$\cos \Psi$	0
z_o	0	0	1

Thus a vector $x_1 \vec{i}_1 + y_1 \vec{j}_1 + z_1 \vec{k}_1$ in the x_1, y_1, z_1 system has the scalar components

$$x_o = x_1 \cos \Psi - y_1 \sin \Psi$$

$$y_o = x_1 \sin \Psi + y_1 \cos \Psi$$

$$z_o = z_1$$

in the (x_o, y_o, z_o) system. Expressed in matrix notation these equations may be written as

$$\begin{vmatrix} x_0 \\ y_0 \\ z_0 \end{vmatrix} = \begin{vmatrix} \cos\Psi & -\sin\Psi & 0 \\ \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ y_1 \\ z_1 \end{vmatrix} \qquad [11a]$$

Thus Equation [11a] represents the first rotation.

ii. Rotate the (x_1, z_1) axes about the y_1 axis through an angle of pitch Θ to (x_2, y_2, z_2). This may be expressed in matrix notation by

$$\begin{vmatrix} x_1 \\ y_1 \\ z_1 \end{vmatrix} = \begin{vmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{vmatrix} \begin{vmatrix} x_2 \\ y_2 \\ z_2 \end{vmatrix} \qquad [11b]$$

iii. Rotate the (y_2, z_2) axes about the x_2 axis through an angle of roll Φ to (x,y,z) the actual orientation of the boat. This may be expressed by

$$\begin{vmatrix} x_2 \\ y_2 \\ z_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} \qquad [11c]$$

Combining Equations [11a], [11b], [11c] gives for the resultant of all three rotations

$$\begin{vmatrix} x_0 \\ y_0 \\ z_0 \end{vmatrix} = \begin{vmatrix} \cos\Psi & -\sin\Psi & 0 \\ \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} \qquad [12]$$

By carrying out the operations indicated by Equation [12], the direction cosines between the x,y,z axes and the x_o, y_o, z_o axis are obtained. These are tabulated below.

	x	y	z
x_o	$\cos\Theta \cos\Psi$	$-\cos\Phi \sin\Psi + \sin\Theta \sin\Phi \cos\Psi$	$\sin\Phi \sin\Psi + \sin\Theta \cos\Phi \cos\Psi$
y_o	$\cos\Theta \sin\Psi$	$\cos\Phi \cos\Psi + \sin\Theta \sin\Phi \sin\Psi$	$-\sin\Phi \cos\Psi + \sin\Theta \cos\Phi \sin\Psi$
z_o	$-\sin\Theta$	$\cos\Theta \sin\Phi$	$\cos\Theta \cos\Phi$

Resolving the U, V, W velocity components in the x_o, y_o, z_o directions gives for the velocity components of the center of gravity of the boat in fixed coordinates.

$$\frac{dx_o}{dt} = U \cos\Theta \cos\Psi + V (\sin\Theta \sin\Phi \cos\Psi - \cos\Phi \sin\Psi) + W (\sin\Phi \sin\Psi + \sin\Theta \cos\Phi \cos\Psi)$$

$$\frac{dy_o}{dt} = U \cos\Theta \sin\Psi + V (\cos\Phi \cos\Psi + \sin\Theta \sin\Phi \sin\Psi) + W (\sin\Theta \cos\Phi \sin\Psi - \sin\Phi \cos\Psi)$$

$$\frac{dz_o}{dt} = -U \sin\Theta + V \cos\Theta \sin\Phi + W \cos\Theta \cos\Phi \qquad [13]$$

To obtain x_o y_o z_o from integration of these equations, as a function of time, in the most general case would clearly require a very considerable effort. However, for most problems involving the motions of hydrofoil boats these equations may be considerably simplified as will be seen later.

4. Angular Orientation of the Hydrofoil Boat

The angular orientation of the hydrofoil boat (Θ,Φ,Ψ) may be expressed in terms of the angular velocity components (P,Q,R) by expressing the angular velocity of the boat in terms of the angles (Θ,Φ,Ψ). Thus the resultant rotation of the boat from orientation (Θ,Φ,Ψ) to orientation $(\Theta+d\Theta,\Phi+d\Phi,\Psi+d\Psi)$ in time dt, using the same order and axes of rotation as in the previous section, may be represented by

$$\vec{d\Omega} = (\Psi + d\Psi)\,\vec{k}_o + (\Theta+d\Theta)\,\vec{j}\,'_1 + (\Phi+d\Phi)\,\vec{i}\,' - \Psi\,\vec{k}_o - \Theta\,\vec{j}_1 - \Phi\,\vec{i}$$

where the subscripts indicate the axes along which the unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ are taken and $\vec{j}\,'_1 \rightarrow \vec{j}_1$ and $\vec{i}\,' \rightarrow \vec{i}$ as $dt \rightarrow 0$

$$\vec{d\Omega} \doteq d\Psi\,\vec{k}_o + d\Theta\,\vec{j}_1 + d\Phi\,\vec{i}$$

and

$$\vec{\omega} = \frac{\vec{d\Omega}}{dt} = \dot{\Psi}\,\vec{k}_o + \dot{\Theta}\,\vec{j}_1 + \dot{\Phi}\,\vec{i} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k} \qquad [14]$$

Resolving $\dot{\Psi}\,\vec{k}_o$ and $\dot{\Theta}\,\vec{j}_1$ along the body axes x,y,z by the use of Equations [11b], [11c] and [12] gives

$$\begin{aligned} P &= \dot{\Phi} - \dot{\Psi}\sin\Theta \\ Q &= \dot{\Theta}\cos\Phi + \dot{\Psi}\cos\Theta\sin\Phi \\ R &= \dot{\Psi}\cos\Theta\cos\Phi - \dot{\Theta}\sin\Phi \end{aligned} \qquad [15]$$

Solving Equation 15 for $\dot{\Theta}$, $\dot{\Phi}$ and $\dot{\Psi}$ gives

$$\begin{aligned} \dot{\Theta} &= Q\cos\Phi - R\sin\Phi \\ \dot{\Phi} &= P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta \\ \dot{\Psi} &= (Q\sin\Phi + R\cos\Phi)\sec\Theta \end{aligned} \qquad [16]$$

Integration of Equation [16] in general would require a complicated numerical procedure. However, for many problems it will be shown later that these equations may be considerably simplified without serious error.

5. Choice of Axes Fixed in the Boat

The equations derived in the preceding sections are valid for any set of orthogonal axes fixed in the boat with origin at the center of gravity, O. However, depending on the problem to be solved there is generally one orientation that is to be preferred to all others. Since it seems fair to assume that all hydrofoil boats have a vertical plane of symmetry, it is generally desirable to make this plane coincide with the xz plane of the body axes. This makes the two products of inertia I_{yz} and I_{xy} equal to zero, thus simplifying Equation [4]. This leaves the directions of the x and z axes to be selected.

Principal Axes - If the x and z axes are taken to coincide with the principal axes of the boat, then the product of inertia I_{zx} also becomes zero further simplifying Equation [4]. These axes are called the principal axes of the boat.

Stability Axes - If the x-axis is taken in the direction of horizontal steady, straight flight then the reference velocity components V and W are zero, thus simplifying the equations of motion. The z-axis will always be taken as positive downward. These axes are referred to as stability axes. In general, these axes do not coincide with the principal axes and will obviously be different for different steady flight trim conditions. The moments and products of inertia relative to these axes I_x, I_z and I_{zx} will therefore depend on the trim conditions. If ϵ represents the angle between the principal x_a-axis and the stability x-axis

and the principal moments of inertia I_{xa} and I_{za} for the boat are known then

$$
\begin{aligned}
I_x &= I_{xa}\cos^2 \epsilon + I_{za} \sin^2 \epsilon \\
I_z &= I_{xa}\sin^2 \epsilon + I_{za} \cos^2 \epsilon \\
I_{zx} &= \tfrac{1}{2} (I_{xa} - I_{za}) \sin 2 \epsilon
\end{aligned}
\qquad [17]
$$

Body Axes - When the x-axis is taken parallel to some fixed reference line in the boat, such as the line from which the trim angle is measured, the resulting set of axes are called body axes. The hydrodynamic force characteristics of the boat are usually measured with respect to these axes. It will be seen later that it is desirable to write the stability equations in the stability axes system. Therefore it will in general be necessary to transform all the force and moment components into the stability axes system.

6. Equations of Motion of the Control Systems

We shall assume that each control system is made up of a control surface or surfaces (rudder, elevator or ailerons) assumed to be rigid, and a mechanical linkage of rigid elements connected to a power source (manual and/or hydraulic, electrical, etc.). Each system is assumed to have one degree of freedom for which a convenient generalized coordinate is the control surface angle. The motion of the control surface will be determined by the moment applied by the power source F_c, the hydrodynamic moment on the control surface H_c, the inertial forces of the system and the gravity forces. In most cases the gravity forces are reasonably constant and not important so that they will not be included in the following analysis.

The forces from the power source are considered to be made up of a manual control force J_c and a power control force P_c which result in an applied moment on the control surface F_c. This moment is related to J_c and P_c by the linkage system. Thus if the displacements due to J_c and P_c are δ_{cJ} and δ_{cP} respectively and this results in a rotation $\delta(\delta_c)$ of the control surface then the work done by P_c and J_c neglecting friction are given as

$$F_c \ \delta(\delta_c) = P_c \ \delta_{c_P} + J_c \ \delta_{c_J}$$

or

$$F_c = P_c \frac{d_{cP}}{d\delta_c} + J_c \frac{d_{cJ}}{d\delta_c} \qquad [18]$$

The inertial force of a mass element δ_m of the control system is

$$\delta\vec{F}' = -\delta_m \vec{a} = -\delta_m \left[\frac{\delta^2 \vec{r}}{\delta t^2} + \vec{a}_c + \frac{\delta \vec{\omega}}{\delta t} \times \vec{r} + 2 \vec{\omega} \times \frac{\delta \vec{r}}{\delta t} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \right] \qquad [19]$$

where the operator $\frac{\delta}{\delta t}$ is defined by Equation [5] and $\vec{a}$ is the acceleration of δ_m (see Reference 1) expressed in terms of $\vec{r}$, the position vector of δ_m in the boat axes. The term

$$\vec{a}_c = \frac{\delta \vec{U}_o}{\delta t} + \vec{\omega} \times \vec{U}_o$$

is the acceleration of the center of gravity of the boat. Only that component of $\delta\vec{F}'$ that acts in the direction in which the mass element is free to move relative to the boat axes will contribute to the motion of the control system. The work done by $\delta\vec{F}'$ during

a virtual displacement $\delta \vec{r}$ of the mass element is

$$\delta \vec{F}' \cdot \delta \vec{r} = -\delta_m \delta r \frac{\delta^2 r}{\delta t^2} + \delta_m \left[\vec{a}_c + \frac{\delta \vec{\omega}}{\delta t} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \right] \cdot \delta \vec{r} \qquad [20]$$

since $\delta \vec{r}$ is colinear with $\frac{\delta \vec{r}}{\delta t}$ and $\frac{\delta^2 \vec{r}}{\delta t^2}$ and the Coriolis Force $\delta_m (2\, \vec{\omega} \times \frac{\delta \vec{r}}{\delta t})$ is perpendicular to $\delta \vec{r}$. The first term on the right of Equation [20] is the inertial work on δ_m due to its acceleration relative to body axes while the remaining terms are due to the acceleration of the boat itself. Since the control system has a single degree of freedom, the magnitude of $\delta \vec{r}$ will be directly related to the rotation of the control surface $\delta(\delta_c)$ and

$$\delta F \cdot (\delta_c) = -\left\{ \delta_m \left(\frac{\delta r}{\delta(\delta_c)} \right)^2 \frac{\delta^2 \delta_c}{\delta t^2} + \delta_m \left[\vec{a}_c + \frac{\delta \vec{\omega}}{\delta t} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \right] \cdot \frac{d \vec{r}}{d(\delta_c)} \right\} \delta(\delta_c) \qquad [21]$$

where δF is the inertial moment at the control surface due to δ_m. The total inertial moment is obtained by summing up over all mass elements. Thus the total inertial moment at the control surface is

$$F = -I_c \frac{d^2 \delta_c}{dt^2} + F_i \qquad [22]$$

where $I_c \left(= \sum \left(\frac{dr}{d(\delta_c)} \right)^2 \delta_m \right)$ is the effective moment of inertia of the control system at the control surface and

$$F_i = -\sum \delta_m \frac{d\vec{r}}{d(\delta_c)} \cdot \left[\vec{a}_c + \frac{\delta\vec{\omega}}{\delta t} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \right]$$

$$= -\sum \delta_m \frac{d\vec{r}}{d(\delta_c)} \cdot \Big\{ [a_{c_x} + (\dot{Q} + PR)z + (QP-\dot{R})y + (Q^2+R^2)x]\vec{i}$$

$$+[a_{c_y} + (\dot{R}+PQ)x+(RQ-\dot{P})z-(P^2+R^2)y]\ \vec{j}$$

$$+[a_{c_z} + (\dot{P}+QR)y+(PR-\dot{Q})x-(P^2+Q^2)z]\ \vec{k} \Big\} \qquad [23]$$

Although for a given control system the second term in Equation [22] may be computed, this would represent a considerable effort except for very simple systems. For the present purpose, we will neglect the effect of the control linkages in the computation of this term, and determine the effect of the motion of the boat on the control surface only. The control surface is approximated by a thin lamina.

The Elevator System - Figure 2 represents an elevator free to rotate about the swept hinge lines shown. It is assumed that the plane of the elevator is parallel to the xy plane of the boat axes and located a distance ℓ_e aft of the center of gravity and a distance z_e below. A small mass element δ_m of the elevator is located a normal distance ρ_e from the hinge line. A positive rotation δ_e of the elevator is considered positive for the trailing edge down. Thus $\frac{d\vec{r}}{d(\delta_e)} = \rho_e \vec{k}$ and the scalar components of the position vector of δ_m is given by

$$x = -(\ell_e + |y'| \sin \Lambda_h + \rho_e \cos \Lambda_h),\ y = y \text{ and } z = z_e\ .$$

Making the above substitutions in Equation [23] gives for this term alone

$$F_i = -\int [a_{cz} + (\dot{P}+QR)\, y - (PR-\dot{Q})\, (\ell_e + |y'| \sin \Lambda_h + \rho_e \cos \Lambda_h)$$

$$- (P^2 + Q^2)\, z_e]\, \rho_e \, dm \,.$$

The second integral is zero for a symmetrical elevator. Hence

$$F_i = - m_e e_e \left[a_{cz} - (P^2 + Q^2) z_e \right] - P_{ex}\, (PR-\dot{Q}) \qquad [24]$$

where $P_{ex} = \int (\ell_e + |y'| \sin \Lambda_h + \rho_e \cos \Lambda_h)\, \rho_e \, dm$

m_e is the mass of both elevators

$e_e \; (= \int \frac{\rho_e dm}{m_e})$ is the eccentricity of the elevator mass center.

Equation [24] represents the inertial force on the elevator due to the acceleration of the boat. This effect is known as the inertial coupling of the elevator. For a dynamically balanced elevator each term on the right of Equation [24] must be zero. This condition is determined by the following consideration. One of the required conditions is that

$$P_{ex} = \int (\ell_e + |y'| \sin \Lambda_h + \rho_e \cos \Lambda_h)\, \rho_e \, dm$$

$$= \ell_e m_e e_e + \sin \Lambda_h \int \rho_e |y'| \, dm + \cos \Lambda_h \int \rho_e^2 \, dm = 0$$

With the other required condition that $e_e = 0$, the sweep angle of the hinge line for dynamic balance is given by

$$\sin \Lambda_h \int \rho_e |y'| \, dm + \cos \Lambda_h \int \rho_e^2 \, dm = 0 \qquad [25]$$

It therefore appears that by statically balancing ($e_e = 0$) the elevator about the hinge line, the sweep of which is given by

Equation [25], it is possible in principle to dynamically balance the elevator. In practice, however, this might be difficult to achieve.

Substituting Equation [24] in Equation [22] and adding the hinge moment applied by the power source F_e (see Equation [18]) and the hydrodynamic hinge moment H_e, leads to the equation of motion of the elevator system.

$$I_e \frac{d^2\delta_e}{dt^2} + m_e e_e [a_{c_z} - (P^2+Q^2) z_e] + P_{ex}(PR-\dot{Q}) = F_e+H_e \qquad [26]$$

This equation does not include the effect of friction. Where friction is a significant factor the effective friction moment at the hinge must be estimated or measured and added to Equation [18].

The Rudder System - The analysis for the rudder system is the same in principle as for the elevator. Figure 2 shows the starboard rudder on one of the forward struts of a hydrofoil boat. It is assumed to be a thin lamina parallel to the xz plane of the boat. Its hinge line is taken parallel to the z-axis located a distance ℓ forward and y_r to starboard of the boat center of gravity. Therefore we have for an element of mass δ_m a distance ρ_r aft of the hinge line

$$x = \ell_r - \rho_r, \; y = y_r \;, \; z = z \;, \; \frac{d\vec{r}}{d(\delta_r)} = -\rho_r \vec{j}$$

Putting these conditions into Equation [23] for the rudder gives for the inertial moment on the rudder due to boat acceleration

$$F_i = \int [a_{cy} + (\dot{R}+PQ)(\ell_r-\rho_r) + (RQ-\dot{P}) z - (P^2+R^2) y_r] \rho_r \, dm$$

$$= m_r e_r [a_{cy} - (P^2+R^2) y_r] + P_{rx}(\dot{R}+PQ) + P_{rz} (RQ-\dot{P}) \qquad [27]$$

where

$$P_{rx} = \int (\ell_r - \rho_r)\, \rho_r \, dm \text{ is a moment of inertia term}$$

$$P_{rz} = \int \rho_r z \, dm \text{ is a product of inertia term.}$$

For a port rudder the sign of y_r is changed. For a rudder whose hinge line is a distance ℓ aft of the center of gravity then $P_{rx} = -\int (\ell_r + \rho_r)\, \rho_r \, dm$.

The equation of motion of the single rudder system is analagous to that for the elevator system

$$I_r \frac{d^2 \delta_r}{dt^2} - m_r e_r \left[a_{cy} - (P^2+R^2) y_r\right] - P_{rx}(\dot{R}+PQ) - P_{rz}(RQ-\dot{P}) = F_r + H_r \quad [28]$$

If a port rudder at $y = -y_r$ is rigidly coupled in a one to one ratio to the starboard rudder then the term containing y_r in Equation [28] cancels out for the coupled system and the equation of motion becomes

$$I_r \frac{d^2 \delta_r}{dt^2} - 2m_r e_r a_{cy} - 2P_{rx}(\dot{R}+PQ) - 2P_{rz}(RQ-\dot{P}) = F_r + H_{r(\text{starboard})} + H_{r(\text{port})} \quad [29]$$

where I_r is the effective moment of inertia of the whole rudder system (including both rudders in the latter case).

<u>The Aileron System</u> - The aileron system differs from the coupled rudder system in that one control surface moves up while the other moves down. Although these surfaces are sometimes arranged so that the upgoing aileron moves a different angle than the downgoing one it will be assumed here that they move through the same angle. The deflection of the starboard aileron is given by δ_a positive downward and the port aileron by δ_a positive upward.

The ailerons are assumed to be lamina parallel to the xy plane with hinge lines parallel to the y-axis which are located a distance ℓ forward and z_a below the boat center of gravity. Therefore we have for an element of mass δ_m a distance a aft of the hinge line

$$x = \ell_a - \rho_a, y=y, \ z=z_a, \ \frac{d\,\vec{r}}{d(\delta_a)} = \rho_a \vec{k}_{(starboard)}, \ \frac{d\,\vec{r}}{d(\delta_a)} = -\rho_a \vec{k}_{(port)}$$

Putting these conditions into Equation [23] for each control surface and adding gives

$$F_i = -2 \int (\dot{P} + QR) \, |y| \, \rho_a \, dm$$

since the terms in a_{cz}, x and z cancel out. The equation of motion then becomes

$$I_a \frac{d^2(\delta_a)}{dt^2} + 2 P_{ay} (\dot{P} + QR) = H_{a(SB)} - H_{a(Port)} + F_a \qquad [30]$$

where I_a is the effective moment of inertia of the whole aileron system (including both surfaces)

F_a is the generalized aileron control force (see Equation [18])

$P_{ay} = \int |y| \, \rho_a \, dm$.

7. External Forces

The forces and moments on the left hand side of Equations [7] and [8] are the external actions on the boat. These are in general of three kinds, gravitational, aerodynamic and hydrodynamic. The gravitational forces are determined by the orientation of the boat. Unless the boat has an aerodynamic propulsion system the aerodynamic forces will usually be small except for high speed boats in which case wind drag is generally an important factor. The hydrodynamic forces are determined by the elevation, orientation, configuration and motion of the boat, z_o, (Θ, Φ, Ψ), $(\delta_e, \delta_r, \delta_a)$, (U, V, W), (P, Q, R) and

the characteristics of the propulsive system.

From Figure 3 the components of weight in the directions of the axes are readily seen to be

$$X_g = - mg \sin \Theta$$

$$Y_g = mg \cos \Theta \sin \Phi$$

$$Z_g = mg \cos \Theta \cos \Phi$$

Thus the external forces in Equation [7] may be written

$$F_x = X - mg \sin \Theta$$

$$F_y = Y + mg \cos \Theta \sin \Phi \qquad [31]$$

$$F_z = Z + mg \cos \Theta \cos \Phi$$

where X, Y and Z are aerodynamic and hydrodynamic forces (including propulsive forces). Since gravity does not contribute to the moments about the mass center, then (L, M, N) are entirely aerodynamic and hydrodynamic.

II. APPLICATIONS OF EQUATIONS OF MOTION

1. Summary of Equations

The kinematic and dynamic equations of motion of the hydrofoil boat and control systems, derived in the foregoing sections, are collected below for ready reference. The xz plane of the coordinate system is taken in the plane of symmetry of the boat so that $I_{yz} = I_{xy} = 0$. The boat is assumed to be a rigid body which may have any number of rigid spinning constant speed rotors attached to it.

$$X - mg \sin \Theta = m (\dot{U} + QW - RV) \qquad (a)$$

$$Y + mg \cos \Theta \sin \Phi = m (\dot{V} + RU - PW) \qquad (b) \qquad [32]$$

$$Z + mg \cos \Theta \cos \Phi = m (\dot{W} + PV - QU) \qquad (c)$$

$$K = I_x \dot{P} - I_{zx} \dot{R} + QR(I_z - I_y) - I_{zx} PQ + Qh_{zR} - Rh_{yR} \quad \text{(a)}$$

$$M = I_y \dot{Q} + RP(I_x - I_z) + I_{zx}(P^2 - R^2) + Rh_{xR} - Ph_{zR} \quad \text{(b)} \quad [33]$$

$$N = I_z \dot{R} - I_{zx} \dot{P} + PQ(I_y - I_x) + I_{zx} QR + Ph_{yR} - Qh_{xR} \quad \text{(c)}$$

$$P = \dot{\Phi} - \dot{\Psi} \sin \Theta \quad \text{(a)}$$

$$Q = \dot{\Theta} \cos \Phi + \dot{\Psi} \cos \Theta \sin \Phi \quad \text{(b)}$$

$$R = \dot{\Psi} \cos \Theta \cos \Phi - \dot{\Theta} \sin \Phi \quad \text{(c)} \quad [34]$$

$$\dot{\Theta} = Q \cos \Phi - R \sin \Phi \quad \text{(d)}$$

$$\dot{\Phi} = P + Q \sin \Phi \tan \Theta + R \cos \Phi \tan \Theta \quad \text{(e)}$$

$$\dot{\Psi} = (Q \sin \Phi + R \cos \Phi) \sec \Theta \quad \text{(f)}$$

$$\frac{dx_o}{dt} = U \cos \Theta \cos \Psi + V (\sin \Theta \sin \Phi \cos \Psi - \cos \Phi \sin \Psi) + W (\sin \Phi \sin \Psi + \sin \Theta \cos \Phi \cos \Psi) \quad \text{(a)}$$

$$\frac{dy_o}{dt} = U \cos \Theta \sin \Psi + V (\cos \Phi \cos \Psi + \sin \Theta \sin \Phi \sin \Psi) + W (\sin \Theta \cos \Phi \sin \Psi - \sin \Phi \cos \Psi) \quad \text{(b)} \quad [35]$$

$$\frac{dz_o}{dt} = - U \sin \Theta + V \cos \Theta \sin \Phi + W \cos \Theta \cos \Phi \quad \text{(c)}$$

$$F_e + H_e = I_e \ddot{\delta}_e + m_e e_e [a_{cz} - (P^2 + Q^2) z_e] + P_{ex} (PR - \dot{Q}) \quad \text{(a)}$$

$$F_r + H_r = I_r \ddot{\delta}_r - m_r e_r [a_{cy} - (P^2 + R^2) y_r] - P_{rx}(\dot{R} + PQ) - P_{rz}(RQ - \dot{P}) \quad \text{(b)} \quad [36]$$

$$F_a + 2 H_a = I_a \ddot{\delta}_a + 2 P_{ay} (\dot{P} + QR) \quad \text{(c)}$$

When the boat is in steady straight horizontal flight V, $(\dot{U}, \dot{V}, \dot{W})$, (P, Q, R), $\frac{dz_o}{dt}$, $\frac{dy_o}{dt}$, Φ, Ψ, in the boat axes are all equal to zero. Thus relative to boat axes

$$X_s - mg \sin \theta = 0$$

$$Y = 0$$

$$Z_s + mg \cos \theta = 0$$

$$K_s = M_s = N_s = 0$$

$$\frac{dx_o}{dt} = U \cos \theta + W \sin \theta$$

$$\frac{W}{U} = \tan \theta \qquad [37]$$

where the subscript s refers to steady flight conditions.

2. Applications of Equations

Equations [34] clearly represent only three independent equations. Additional equations must be added to Equation [36] when more than one control system of a given type is used. For example if the boat has independent fore and aft elevators, there would be a separate equation for each elevator system. Thus, in general, for n independent control systems Equations [36] consist of n independent equations. Since the aerodynamic and hydrodynamic forces and moments (X, Y, Z), (K, M, N) and H_{c_n} depend only on the sea state and the elevation, orientation, configuration and motion variables z_o, (θ, Φ) δ_{c_n}, (U, V, W) (P, Q, R) they do not represent independent unknowns. Also Equation [35a, b] may be dropped since the remaining equations do not depend on x_o and y_o. Thus Equations [32]-[36] may be reduced in general to 10 + n independent

equations in 10 + 2n unknowns. The unknowns are (U, V, W), (P, Q, R), (Θ, Φ, Ψ), z_o, δ_{c_n} and F_{c_n} where δ_{c_n} and F_{c_n} are the angle and control force on the n-th control system. It is therefore clear that n of these variables must be specified before these equations can be solved and the motion determined. The variables selected and the manner in which they are specified are determined by the problem to be solved.

Stability Problems - Controls Fixed - In these problems the boat is initially in steady, level, flight in a calm sea and it is desired to determine the motion caused by a disturbance of very short duration. The controls are assumed locked in the position for steady, level flight. Since the control angles δ_{c_n} are fixed and known Equations [36] are not needed to determine the boat motion. However, after the boat motion is determined these equations may be used, if desired, to obtain the control forces F_{c_n} necessary to hold the control surfaces in the fixed condition. Since the aerodynamic and hydrodynamic forces do not depend on x_o and y_o, Equations [35 a,b] may also be dropped. The result is a set of ten equations in ten unknowns. These equations are non-linear and difficult to solve. For many practical problems it is sufficient to linearize the equations by dealing with small perturbations from equilibrium flight. In this case the resulting equations are a set of homogeneous linear differential equations with constant coefficients which are readily solved.

Stability Problems - Controls Free - These problems are similar to those discussed above except that in this case the control angles are variable and Equations [36] with the control forces, F_{c_n}, set equal to zero, must be added to the above system

of equations. If the equations are linearized they again become homogeneous and are treated in the same manner as in the fixed control case.

Stability Problems - Automatic Control - In these problems the controls are neither fixed nor free. Even though the pilot input P_{c_n} is zero the generalized control forces are not. They are in general determined by a feedback of the boat response into the input of the control system. All 10 + n equations are in general involved, and the dynamics of the closed loop must be introduced.

Response to Controls - In these problems it is desired to determine the effectiveness of the boat's controls. This is normally done by specifying a particular variation of the control surface angles δ_{c_n} with time, e.g. a step-function input of rudder angle. The equations of motion of the boat then become inhomogeneous equations for (U, V, W), (P, Q, R), (Θ, Φ, Ψ) and z_o. The control-systems equations are not needed but may be used if desired to compute the variations in the control forces F_{c_n} required to produce the maneuver.

Response to Seaway - The response of a boat to a seaway is treated with the same equations as are used for stability. The effect of the seaway introduces new forcing functions on the lifting surfaces and the struts. The equations of motion then become inhomogeneous and the mathematical treatment is similar to that used for the response to controls. For seas in which the motions are large it is necessary to solve the equations without the simplification of linearization. This is in general a formidable task requiring a high speed digital computer and has heretofore been carried out in a few cases and only for simple

foil systems performing longitudinal motions in regular head and following seas. It has been found that when the motions are not very large, linearization of the equations of motion give a reasonably good approximation of the boat motions. It has been customary to use the linearized equations in the study of automatic control systems in relation to the seaway.

Inverse Problems - This class of problems arises when specified variables are selected from those that are normally treated as dependent, (U, V, W) (P, Q, R), (Θ, Φ, Ψ) and z_o. For example it may be required to find the control forces required to produce a certain steady level turn. In this case one might specify the roll angle Φ, the turning radius R_o ($= U_o/\dot{\Psi}$), the elevation of the center of gravity $z_o (= 0)$ and n - 3 control surface angles. Since the motion is assumed steady this would lead to a set of algebraic equations. If, on the other hand, $\dot{\Psi}$ and $\dot{\Phi}$ are specified as functions of time a set of differential equations are obtained. A decided advantage of this method is its ability, in many cases, to deal with non-linear equations.

III. THE LINEARIZED EQUATIONS OF MOTION

1. Derivation - The Dimensional Equations

The solutions to the equations of motion when the motion of the boat is limited to small infinitesmal disturbances of a symmetrical boat from steady equilibrium flight in a calm sea has been extensively studied by many investigators. Although they apply strictly to infinitesmal disturbances only, the results obtained have been found to apply with fair accuracy to finite disturbances which are sufficiently large to make the conclusions drawn of great practical interest.

In the following discussion we shall carry through the details of the linearization for Equation [32a] and [33a] only. The other motions equations are treated in an analogous manner. As stated earlier the aerodynamic and hydrodynamic forces and moments depend on the elevation, orientation, configuration and motion variables. Thus in Equation [32a] and [33a]

$$X = X\ (U,V,W,\dot{U},\dot{V},\dot{W},P,Q,R,\dot{P},\dot{Q},\dot{R},\theta,\Phi,\Psi,\delta_{a_n},\delta_{e_n},\delta_{r_n},z_o) \qquad [38]$$

$$K = K\ (U,V,W,\dot{U},\dot{V},\dot{W},P,Q,R,\dot{P},\dot{Q},\dot{R},\theta,\Phi,\Psi,\delta_{a_n},\delta_{e_n},\delta_{r_n},z_o) \qquad [39]$$

The boat is assumed to be in some equilibrium reference flight condition. The initial reference values of all the variables are denoted by a subscript i, and the small perturbations are indicated as follows: changes in U V W etc. are indicated by small letters, i.e. $U=u_i + u$, $P=p_i + p$, $\theta = \theta_i + \theta$, $z_o = z_i + h$ etc. Changes in the control-surface angles are denoted by (ξ,η,ζ) so that

$$\delta_{a_n} = \delta_{a_{ni}} + \xi_n$$

$$\delta_{e_n} = \delta_{e_{ni}} + \eta_n$$

$$\delta_{r_n} = \delta_{r_{ni}} + \zeta_n \qquad [40]$$

For steady equilibrium the reference values of the variables are taken as constant. Making the appropriate substitutions in Equations [32a] and [33a] yields

$$X - mg\sin(\theta_i+\theta) = m\,[\dot{u}+(q_i+q)(w_i+w)-(r_i+r)(v_i+v)] \qquad [41]$$

$$K = I_x\dot{P} - I_{zx}\dot{r} + (q_i+q)(r_i+r)(I_z-I_y)-I_{zx}(p_i+p)(q_i+q)+(q_i+q)h_{zR} - (r_i+r)h_{yR} \qquad [42]$$

where X and K are given by

$$X = X\ (u_i + u,\ v_i + v,\ \text{----------},\ \delta_{r_{ni}} + \zeta_n,\ z_i + h) \qquad [43]$$

$$K = K\ (u_i + u,\ v_i + v,\ \text{----------},\ \delta_{r_{ni}} + \zeta_n,\ z_i + h) \qquad [44]$$

The Taylor expansion of Equations [43] and [44] is

$$X = [1+(uD_u+vD_v+\text{---}+hD_h) + \frac{(uD_u+vD_v+\text{---}+hD_h)^2}{2!} + \text{---}]\ X(u_i, v_i, \text{--}, z_i)$$

$$K = [1+(uD_u+vD_v+\text{---}+hD_h) + \frac{(uD_u+vD_v+\text{---}+hD_h)^2}{2!} + \text{---}]\ K(u_i, v_i, \text{---}, z_i)$$

Since we are dealing with small perturbations from the reference flight condition, the second and higher order terms may be omitted leaving only the first two terms of each of the above series. Writing the above operators D in their partial derivative notation

$$X = X\ (u_i, v_i\ \text{---},\ z_i) + u \left.\frac{\partial X}{\partial u}\right|_i + v \left.\frac{\partial X}{\partial v}\right|_i + \text{---} + h \left.\frac{\partial X}{\partial h}\right|_i \qquad [45]$$

$$K = K\ (u_i, v_i\ \text{---},\ z_i) + u \left.\frac{\partial K}{\partial u}\right|_i + v \left.\frac{\partial K}{\partial v}\right|_i + \text{---} + h \left.\frac{\partial K}{\partial h}\right|_i \qquad [46]$$

where the subscript i denotes that the partial derivative is taken at the initial equilibrium condition. The use of only first order terms implies the assumption that the forces and moments vary linearly with the disturbance variables for small enough disturbances.

Also in its present form it is assumed that the derivatives depend only on the instantaneous values of the disturbance velocities, control angles, etc. and their derivatives. Despite the success of this last assumption in many types of problems it is in general not sound. It may lead to serious errors in cases where the hydrodynamic forces change very rapidly, as in flight in rough seas at high speeds or when a control is very rapidly displaced. An alternative set of small disturbance equations will be given later which is not subject to these limitations.

On substituting Equations [45] and [46] in Equations [41] and [42] and expanding $\sin(\theta_i + \theta)$ one obtains

$$X_i + u\left.\frac{\partial X}{\partial u}\right|_i + v\left.\frac{\partial X}{\partial v}\right|_i + \text{---} + h\left.\frac{\partial X}{\partial h}\right|_i - mg\,(\sin\theta_i + \theta\cos\theta_i)$$

$$= m\,[\dot{u} + (q_i+q)\,(w_i+w) - (r_i+r)\,(v_i+v)] \qquad [47]$$

$$K_i + u\left.\frac{\partial K}{\partial u}\right|_i + v\left.\frac{\partial K}{\partial v}\right|_i + \text{---} + h\left.\frac{\partial K}{\partial h}\right|_i = I_x\dot{p} - I_{zx}\dot{r} + (q_i+q)(r_i+r)(I_z-I_y)$$

$$- I_{zx}(p_i+p)\,(q_i+q) + (q_i+q)\,h_{zR} - (r_i+r)\,h_{yR} \qquad [48]$$

Since X_i and K_i are equal to the unperturbed terms on the right hand side of Equations [47] and [48] respectively they may be cancelled out. Retaining only first order terms and writing X_u for $\left.\frac{\partial X}{\partial u}\right|_i$ etc. gives

$$(X_{\dot{u}}-m)\,\dot{u} + X_{\dot{v}}\dot{v} + X_{\dot{w}}\dot{w} + X_u u + (X_v+mr_i)v + (X_w-mq_i)w + X_{\dot{p}}\dot{p} + X_{\dot{q}}\dot{q} + X_{\dot{r}}\dot{r} + X_p p$$

$$+ (X_q-mw_i)q + (X_r+mv_i)r + (X_\theta - mg\cos\theta_i)\theta + X_\phi\phi + X_\Psi\Psi + X_{\dot{\xi}}\dot{\xi} + X_{\dot{\eta}}\dot{\eta}$$

$$+ X_{\dot{\zeta}}\dot{\zeta} + X_\xi\xi + X_\eta\eta_\eta + X_\zeta\zeta + X_h h = 0 \qquad [49]$$

and

$$K_{\dot{u}}\dot{u}+K_{\dot{v}}\dot{v}+K_{\dot{w}}\dot{w}+K_u u+K_v v+K_w w+(K_{\dot{p}}-I_x)\dot{p}+K_{\dot{q}}\dot{q} + (K_{\dot{r}}+I_{zx})\dot{r}+(K_p+I_{zx}q_i)p$$
$$+[K_q-r_i(I_z-I_y)-h_{zR}+I_{zx}p_i]q+[K_r-q_i(I_z-I_y)+h_{yR}]r+K_\theta\theta+K_\phi\phi+K_\Psi\Psi+$$
$$K_{\dot{\xi}}\dot{\xi}+K_{\dot{\eta}}\dot{\eta}+K_{\dot{\zeta}}\dot{\zeta}+K_\xi\xi+K_\eta\eta+K_\zeta\zeta+K_h h = 0 \qquad [50]$$

The remaining equations in the set of Equations [32] - [36] may be determined in an analagous manner. These equations give the perturbation motions from any equilibrium flight condition.

For the very important case of straight, level, symmetrical flight these equations may be considerably simplified. For a truly symmetrical boat it is clear that the side force Y, rolling moment K, yawing moment N, and aileron and rudder hinge moments H_r and H_a will be zero. Thus the derivatives of the asymmetric forces and moments with respect to the symmetric variables u, w, q, η, θ, h are all zero. Furthermore $v_i = p_i = q_i = r_i = \phi_i = 0$. We will use stability axes so that $w_i = 0$ and u_i is then the reference boat speed U_o. Since we are considering level flight $\theta_i = 0$. In addition the following approximations will be made.

1. We may neglect the derivatives of the symmetric forces and moments X, Z, M, H_e with respect to the asymmetric variables v, p, r, ϕ, ξ, ζ.

2. We may neglect all the acceleration derivatives except the following $Z_{\dot{w}}$, $Z_{\dot{q}}$, $M_{\dot{w}}$, $M_{\dot{q}}$, $H_{e_{\dot{w}}}$, $H_{e_{\dot{q}}}$, $Y_{\dot{v}}$, $Y_{\dot{p}}$, $Y_{\dot{r}}$, $K_{\dot{v}}$, $K_{\dot{p}}$, $K_{\dot{r}}$, $N_{\dot{v}}$, $N_{\dot{p}}$, $N_{\dot{r}}$, $H_{a_{\dot{p}}}$, $H_{r_{\dot{p}}}$, $H_{r_{\dot{v}}}$, $H_{r_{\dot{r}}}$. In practice it will turn out that most of the above derivatives will often be negligible.

3. We may neglect H_{a_h}, H_{e_h} and H_{r_h} and the angular momentum of the rotors.

When the above simplifications are made the linearization of Equations [32] - [36] for calm sea operation results in Equations [51] and [52] (page 32). The effects of a seaway are taken into account by simply adding the hydrodynamic forces and moments imposed by the sea to the right hand side of these equations. In general these would be a function of time and the perturbation variables of the boat. However, useful results have generally been obtained by neglecting the dependence of the wave forces on the boat perturbations as long as the boat motions are not too violent. In this case the wave forces and moments are computed as if the boat were moving in steady level flight and appropriately added to the right hand sides of Equations [51] and [52].

Due to the simplifying assumptions that were made in the previous section, Equations [51] and [52] have become uncoupled. Equations [51] are a function only of u, w, q, θ, h, x, and η, the longitudinal variables. Solution of this set of equations gives the perturbed longitudinal motions of the boat. Equations [52] contain only the variables v, p, r, Φ, Ψ, y, ξ and ζ. Solutions of these equations gives the perturbed lateral motions of the boat.

In the computation of longitudinal motions it is often convenient to eliminate w from Equations [51a,b,c]. Substituting Equation [51f] into Equations [51a,b,c] gives

$$(mD-X_u)u-(X_w D+X_h)h-(X_q D+X_w U_o+X_\theta-mg)\,\theta = 0 \qquad (a)$$

$$-Z_u u+[(m-Z_{\dot{w}})D^2-Z_w D-Z_h]h-[Z_{\dot{q}}D^2+(Z_q+Z_{\dot{w}}U_o)D+Z_w U_o+Z_\theta]\theta-Z_\eta\eta = 0 \qquad (b)$$

$$-M_u u-(M_{\dot{w}}D^2+M_w D+M_h)h+[(I_y-M_{\dot{q}})D^2-(M_q+M_{\dot{w}}U_o)D-(M_w U_o+M_\theta)]\theta$$
$$-(M_\eta D+M_\eta)\eta = 0 \qquad (c)\ [53]$$

LONGITUDINAL EQUATIONS

$$(mD-X_u)u-X_w w-[X_q D+(X_\theta-mg)]\,\theta - X_h h = 0 \quad \text{(a)}$$

$$-Z_u u+[(m-Z_{\dot w})D-Z_w]w-[Z_{\dot q}D^2+(Z_q+mU_o)D+Z_\theta]\theta-Z_h h-Z_\eta \eta = 0 \quad \text{(b)}$$

$$-M_u u-(M_{\dot w}D+M_w)w+[(I_y-M_{\dot q})D^2-M_q D-M_\theta]\theta-M_h h-(M_{\dot\eta}D+M_\eta)\eta = 0 \quad \text{(c)}$$

$$-He_u u-[(He_{\dot w}-m_e e_e)D+He_w]w-[(He_{\dot q}+P_{ex})D^2+(He_q+m_e e_e U_o)D]\theta+(I_e D^2-He_{\dot\eta}D-He_\eta)\eta=\Delta F_e \quad \text{(d)}$$

$$q = \dot\theta \quad \text{(e)}$$

$$\dot h = w - U_o\theta \quad \text{(f)}$$

$$\frac{dx_o}{dt} = u \quad \text{(g)} \quad [51]$$

LATERAL EQUATIONS

$$[(m-Y_{\dot v})D-Y_v]v-[Y_{\dot p}D^2+Y_p D+(Y_\phi+mg)]\phi-[Y_{\dot r}D+(Y_r-mU_o)]r-Y_\zeta\zeta = 0 \quad \text{(a)}$$

$$-(K_{\dot v}D+K_v)v+[(I_x-K_{\dot p})D^2-K_p D-K_\phi]\phi-[(K_{\dot r}+I_{zx})D+K_r]r-(K_{\dot\xi}D+K_\xi)\xi-(K_{\dot\zeta}D+K_\zeta)\zeta=0 \quad \text{(b)}$$

$$-(N_{\dot v}D+N_v)v-[(N_{\dot p}+I_{zx})D^2+N_p D+N_\phi]\phi+[(I_y-N_{\dot r})D-N_r]r-N_\xi\xi-(N_{\dot\zeta}D+N_\zeta)\zeta=0 \quad \text{(c)}$$

$$+2[(P_{ay}-H_{a\dot p})D^2-H_{ap}D]\phi-2H_{ar}r+[I_a D^2-2H_{a\dot\xi}D-2H_{a\xi}]\xi = \Delta F_a \quad \text{(d)}$$

$$-[(H_{r\dot v}+m_r e_r)D+H_{r_v}]v+[(P_{rz}-H_{r\dot p})D^2-H_{r_p}D]\phi-[(H_{r\dot r}+P_{rx})D+m_r e_r U_o+H_{r_r}]r$$
$$+(I_r D^2-H_{r\dot\zeta}D-H_{r_\zeta})\zeta = \Delta F_r \quad \text{(e)}$$

$$p = \dot\phi\ ;\quad r = \dot\Psi\ ;\ \frac{dy_o}{dt} = U_o\,\Psi + v \quad \text{(f)} \quad [52]$$

2. Dimensionless Equations

Equations [51a,b,c] and [52a,b,c] are non-dimensionalized by dividing the force equations by $\frac{1}{2}\rho A U_o^2$ and the moment equations by $\frac{1}{2}\rho A \ell U_o^2$. The quantity A is a reference area usually taken as the projected area of the foil system on the xy plane of the body axes and ℓ is a reference length usually the distance between two convenient points on the forward and aft hydrofoil systems. For the sake of uniformity it is suggested that the longitudinal distance between forward-most points of the forward and aft foil systems be used.

Linear velocities are non-dimensionalized by dividing by U_o and angular velocities by U_o/ℓ. The mass terms are divided by $\frac{1}{2}\rho A \ell$. The control surface equations are non-dimensionalized by dividing the hinge moment H_c and control force F_c by $\frac{1}{2}\rho A_c \bar{c}_c U_o^2$ where A_c is the area of that portion of the control surface and tab area that lies aft of the hinge line and $\bar{c}_c$ is the mean chord of the same portion of the control surface and tab area. The non-dimensional quantities are indicated by the prime symbol. Thus, for example

$$X_u' \, u' = \frac{X_u}{\frac{1}{2}\rho A U_o} \cdot \frac{u}{U_o}$$

$$K_p' \, p' = \frac{K_p}{\frac{1}{2}\rho \ell^2 A U_o} \cdot \frac{p\ell}{U_o}$$

$$H_{e_u}' \, u' = \frac{H_{e_u}}{\frac{1}{2}\rho \bar{c}_e A_e U_o} \cdot \frac{u}{U_o}$$

$$I_{zx}' = \frac{I_{zx}}{\frac{1}{2}\rho A \ell^3} \; ; \quad m' = \frac{m}{\frac{1}{2}\rho A \ell} \; ; \quad m'g' = \frac{mg}{\frac{1}{2}\rho U_o^2 A} \doteq C_{L_o}$$

$$D' = \frac{d}{dt'} = \frac{d}{d\left(\frac{U_o t}{\ell}\right)}$$

$$U_o' = \frac{U_o}{U_o} = 1 \; ; \quad H_{r\dot{\zeta}}' \; \dot{\zeta}' = \frac{H_{r\dot{\zeta}}}{\frac{1}{2}\rho \overline{c}_e A_e U_o \ell} D'\zeta$$

A summary is given in Table I.

TABLE I.

THE NON-DIMENSIONAL SYSTEM

DIMENSIONAL QUANTITY	DIVISOR	NON-DIMENSIONAL QUANTITY
$X \; Y \; Z$	$\frac{1}{2}\rho U_o^2 A$	$X' \; Y' \; Z'$
$K \; M \; N$	$\frac{1}{2}\rho U_o^2 A \ell$	$K' \; M' \; N'$
$H_a \; H_e \; H_r$	$\frac{1}{2}\rho U_o^2 A_c \overline{c}_c$	$H_a' \; H_e' \; H_r'$
$F_a \; F_e \; F_r$	$\frac{1}{2}\rho U_o^2 A_c \overline{c}_c$	$F_a' \; F_e' \; F_r'$
$u \; v \; w$	U_o	$u' \; v' \; w'$
$p \; q \; r$	U_o/ℓ	$p' \; q' \; r'$
$D = \frac{d}{dt}$	U_o/ℓ	D'
$\dot{u} \; \dot{v} \; \dot{w}$	U_o^2/ℓ	$D'u' \; D'v' \; D'w'$
$\dot{p} \; \dot{q} \; \dot{r}$	U_o^2/ℓ^2	$D'p' \; D'q' \; D'r'$
$\dot{\xi} \; \dot{\eta} \; \dot{\zeta}$	U_o/ℓ	$D'\xi \; D'\eta \; D'\zeta$
m	$\frac{1}{2}\rho A \ell$	m'
I_x, I_y, I_z, I_{zx}	$\frac{1}{2}\rho A \ell^3$	$I_x', I_y', I_z', I_{zx}'$
$m_a, \; m_e, \; m_r$	$\frac{1}{2}\rho A_c \ell$	$m_a' \; m_e' \; m_r'$
$e_a, \; e_e, \; e_r$	$\overline{c}_c$	$e_a' \; e_e' \; e_r'$
$P_{ex} \; P_{ay} \; P_{rz}$	$\frac{1}{2}\rho \overline{c}_c \ell^2$	$P_{ex}' \; P_{ay}' \; P_{rz}'$
$I_a \; I_e \; I_r$	$\frac{1}{2}\rho \overline{c}_c A_c \ell^2$	$I_a' \; I_e' \; I_r'$

DIMENSIONLESS LONGITUDINAL EQUATIONS [54]

$$(m'D'-X_u')\,u'-X_w'w' - [X_q'D' + (X_\theta'-C_{L_o})]\,\theta - X_h'h' = 0 \qquad (a)$$

$$-Z_u'u' + [(m'-Z_{\dot{w}}')D'-Z_w']w' - [Z_{\dot{q}}'D'^2 + (Z_q'+m')D'+Z_\theta']\,\theta - Z_h'h' - Z_\eta'\eta = 0 \qquad (b)$$

$$-M_u'u' - (M_{\dot{w}}'D'+M_w')w' + [(I_y'-M_{\dot{q}}')D'^2 - M_q'D' - M_\theta']\,\theta - M_h'h' - (M_{\dot{\eta}}'D' + M_\eta')\,\eta = 0 \qquad (c)$$

$$-H_{e_u}'u' - [(H_{e_{\dot{w}}}' - m_e'e_e')D' + H_{e_w}']w' - [(H_{e_{\dot{q}}}'+P_{ex}')D'^2 + (H_{e_q}' + m_e'e_e')D']\,\theta + (I_e'D'^2 - H_{e_{\dot{\eta}}}'D' - H_{e_\eta}')\,\eta = \Delta F_e' \qquad (d)$$

$$q' = \dot{\theta}' \; ; \quad \dot{h}' = w' - \theta \; ; \quad D'x_o' = u' \qquad (e)$$

DIMENSIONLESS LATERAL EQUATIONS [55]

$$[(m'-Y_{\dot{v}}')D'-Y_v']v' - [Y_{\dot{p}}'D'^2 + Y_p'D' + (Y_\phi' + C_{L_o})]\,\phi - [Y_{\dot{r}}'D' + (Y_r'-m')]r' - Y_\zeta'\zeta = 0 \qquad (a)$$

$$-(K_{\dot{v}}'D'+K_v')v' + [(I_x'-K_{\dot{p}}')D'^2 - K_p'D' - K_\phi']\,\phi - [(K_{\dot{r}}'+I_{zx}')D'+K_r']r' - (K_{\dot{\xi}}'D'+K_\xi')\,\xi - (K_{\dot{\zeta}}'D'+K_\zeta')\,\zeta = 0 \qquad (b)$$

$$-(N_{\dot{v}}'D'+N_v')v' - [(N_{\dot{p}}'+I_{zx}')D'^2 + N_p'D' + N_\phi']\,\phi + [(I_z'-N_{\dot{r}}')D' - N_r']r' - N_\xi'\xi - (N_{\dot{\zeta}}'D'+N_\zeta')\,\zeta = 0 \qquad (c)$$

$$2\,[(P_{ay}' - H_{a_{\dot{p}}}')D'^2 - H_{a_p}'D']\,\phi - 2H_{a_r}'r' + [I_a'D'^2 - 2H_{a_{\dot{\xi}}}'D' - 2H_{a_\xi}']\,\xi = \Delta F_a' \qquad (d)$$

$$-[(H_{r_{\dot{v}}}' + m_r'e_r')D' + H_{r_v}']v' + [(P_{rz}'-H_{r_{\dot{p}}}')D'^2 - H_{r_p}'D']\,\phi - [(H_{r_{\dot{r}}}' + P_{rx}')D' + m_r'e_r' + H_{r_r}']\,r'$$

$$+ (I_r'D'^2 - H_{r_{\dot{\zeta}}}'D' - H_{r_\zeta}')\,\zeta = \Delta F_r' \qquad (e)$$

$$p' = D'\Phi \; ; \quad r' = D'\Psi \; ; \quad D'y_o' = \Psi + v' \qquad (f)$$

ALTERNATE DIMENSIONLESS LONGITUDINAL EQUATIONS [56]

$$(m'D'-X_u')\,u' - (X_w'D'+X_h')\,h' - (X_q'D' + X_w' + X_\theta' - C_{L_o})\,\theta = 0 \qquad (a)$$

$$-Z_u'u' + [(m'-Z_{\dot{w}}')D'^2 - Z_w'D' - Z_h']h' - [Z_{\dot{q}}'D'^2 + (Z_q' + Z_{\dot{w}}')D' + Z_w' + Z_\theta']\,\theta - Z_\eta'\eta = 0 \qquad (b)$$

$$-M_u'u' - (M_{\dot{w}}'D'^2 + M_w'D' + M_h')h' + [(I_y' - M_{\dot{q}}')D'^2 - (M_q' + M_{\dot{w}}')D' - (M_w' + M_\theta')]\,\theta - (M_\eta'D' + M_\eta')\,\eta = 0 \qquad (c)$$

Equations [51] and [52] may be written in dimensionless form simply by placing primes in the manner above indicated on all the terms and making the substitutions $U_o' = 1$ and $m'g' = C_{L_o}$. The resulting dimensionless equations are sometimes written without the prime symbols for the sake of simplicity in notation. By making the above substitutions in Equations [51], [52], [53] the dimensionless Equations [54], [55], [56] are obtained (see page 35).

3. Hydrodynamic Transfer Functions

In the previous sections the hydrodynamic derivatives were written in quasistatic form. For example the change in the dimensionless Z'-force due to a perturbation w' was written as

$$Z_1' = (Z_w' + Z_{\dot{w}}' D')\, w' \qquad [57]$$

This is correct as long as w' is a slowly changing function of time. However, when rapid motions are involved the above equation does not correctly represent the force. In general however the relation between Z_1' and w' may be assumed to be representable by a linear differential or integral equation of more complex form. The hydrodynamic transfer function relating w' and Z_1' is defined as the ratio of the Laplace Transforms of Z_1' and w'

$$G_{Zw}(s) = \frac{\overline{Z_1'(s)}}{\overline{w'(s)}} \qquad [58]$$

where $\overline{Z_1'(s)} = \int_{t'=0}^{\infty} e^{-st'}\, Z_1'(t')\, dt'$; etc. (See Reference 17)

The transfer function is unique if $w'(t) = Z_1'(t) = 0$ for $t' < 0$. This requires that we deal with motions for which the initial conditions are those of the reference flight conditions. Because of

the assumed linearity between Z' and the perturbations we may write for the Laplace Transform of the change in Z' force due to perturbations u', w', q', etc.

$$\overline{Z} = G_{Zu}\,\overline{u} + G_{Zw}\,\overline{w} + G_{Zq}\,\overline{q} + --- \qquad [59]$$

By taking the Laplace Transform of Equations [54] and [55] and replacing terms like $(Z_{\dot{w}}'D' + Z_w')w'$ by $G_{Zw}\,\overline{w}$, we obtain Equation [62] and Equation [63]. If the motion is a slowly changing function of time, as is often the case in stability calculations, then Equation [57] would be a sufficiently good approximation to Z_1' and $G_{Zw} = Z_w' + Z_{\dot{w}}'\,s$. However, when an oscillatory mode of high frequency exists it may be necessary to obtain a more precise value for G_{Zw}. The use of stability derivatives implies that a more accurate approximation to Z_1' is given by

$$Z_1' = (Z_w' + Z_{\dot{w}}'D' + Z_{\ddot{w}}'D'^2 + ---)w' \qquad [60]$$

and the resulting transfer function would be

$$G_{Zw} = Z_w' + Z_{\dot{w}}'s + Z_{\ddot{w}}'s^2 + --- \qquad [61]$$

Although an infinite series such as this can sometimes give a correct value for G_{Zw} there are other times when the series is not convergent (see Reference 2).

LONGITUDINAL EQUATIONS

$$(m's - G_{Xu})\,\overline{u} - G_{Xw}\,\overline{w} - (G_{X\theta} - C_{L_o})\,\overline{\theta} - G_{Xh}\,\overline{h} = 0 \quad \text{(a)}$$

$$- G_{Zu}\,\overline{u} + (m's - G_{Zw})\,\overline{w} - (G_{Z\theta} + m's)\,\overline{\theta} - G_{Zh}\,\overline{h} - G_{Z\eta}\,\overline{\eta} = 0 \quad \text{(b)}$$

$$- G_{Mu}\,\overline{u} - G_{Mw}\,\overline{w} + (I_y's^2 - G_{M\theta})\,\overline{\theta} - G_{Mh}\,\overline{h} - G_{M\eta}\,\overline{\eta} = 0 \quad \text{(c)}$$

$$- G_{Hu}\,\overline{u} - (G_{Hw} - m_e'e_e's)\,\overline{w} - [P_{ex}'s^2 + (G_{Hq} + m_e'e_e')s]\theta + (I_e's^2 - G_{H\eta})\eta = \overline{\Delta F_e} \quad \text{(d)} \quad [62]$$

LATERAL EQUATIONS

$$(m's - G_{Yv})\,\overline{v} - (G_{Y\phi} + C_{L_o})\,\overline{\phi} - (G_{Yr} - m')\,\overline{r} - G_{Y\zeta}\,\overline{\zeta} = 0 \quad \text{(a)}$$

$$-G_{Kv}\,\overline{v} + (I_x's^2 - G_{K\phi})\,\overline{\phi} - (I_{zx}'s + G_{Kr})\,\overline{r} - G_{K\xi}\,\overline{\xi} - G_{K\zeta}\,\overline{\zeta} = 0 \quad \text{(b)}$$

$$-G_{Nv}\,\overline{v} - (I_{zx}'s^2 + G_{N\phi})\,\overline{\phi} + (I_z's - G_{Nr})\,\overline{r} - G_{N\xi}\,\overline{\xi} - G_{N\zeta}\,\overline{\zeta} = 0 \quad \text{(c)}$$

$$2\,(P_{ay}'s^2 - G_{Hp_a}s)\,\overline{\phi} - 2G_{Hr_a}\,\overline{r} + (I_a's^2 - 2G_{H\xi_a})\,\overline{\xi} = \overline{\Delta F_a} \quad \text{(d)}$$

$$-(G_{Hv_r} + m_r'e_r's)\,\overline{v} + (P_{rz}'s^2 - G_{Hp_r}s)\,\overline{\phi} - (P_{rx}'s + m_r'e_r' + G_{Hr_r})\overline{r} + (I_r's^2 - G_{H\zeta_r})\overline{\zeta} = \overline{\Delta F_r} \quad \text{(e)} \quad [63]$$

IV. EQUATIONS OF MOTION OF AN ELASTIC HYDROFOIL BOAT

1. Introduction

In the preceding sections the equations of motion of the hydrofoil boat have been derived on the assumption that the structure is perfectly rigid. On this basis the externally applied hydrodynamic forces are put in equilibrium with the hydrodynamic and inertial forces resulting from the translational and rotational motions of the boat as a rigid body. Though this assumption is often valid there is no assurance that this is always so, especially in view of recent tendencies toward lighter construction and higher speeds. Deformation of the hull and foil-strut system may induce additional hydrodynamic forces which affect the overall response of the boat. In addition, if the dynamic response is a result of rapidly applied external forces such as those applied in a rough sea the boat will not only be caused to rotate and translate but structural vibrations may be induced as well. The latter may have a significant effect upon the internal stress distribution of the structure.

Most of the important hydroelastic effects on stability and control may be taken into account simply by modifying the hydrodynamic derivatives. Thus if the frequencies of the rigid boat motions are much smaller than the natural frequencies of the elastic deformation modes then it may be assumed that the hydrodynamic loads change slowly enou so that the deformation of the structure takes place in the same manner as would be the case if the instantaneous load were applied as a static load. Thus a change in load produces a proportional change in structure shape which in turn changes the load. Examples of stability derivatives which take this effect into account are given in References 3 - 8.

When the separation in frequency between the elastic degrees of freedom and the rigid body motions is not large then significant coupling may occur between the two. In that case a dynamic analysis is required. It was seen that the rigid body equations described the motion of the boat in terms of six degrees of freedom: pitch, roll, yaw, surge, sway and heave. When the elastic properties of the boat are taken into account also, this amounts to introducing an infinite number of additional degrees of freedom. In practice however, the elastic deformations may be approximated very closely by a finite number of degrees of freedom. Three methods are described in References 4 and 5 for reducing a continuous system to one with a finite number of degrees of freedom. In the first, the deformation is assumed as a super-position of a finite number of natural modes of the structure. These modes are obtained from the eige values and eigen functions of the homogeneous equations of motion of the unrestrained boat. In the second, the deformation is taken as a super-position of a finite number of assumed mode shapes, and in the third, the deformation is described by the deflections at a number of discrete points on the surface of the structure. This last method is sometimes called the collocation approach and is perhaps to be preferred where automatic computing machines are available. This method is sometimes used in combination with the method of assumed modes. The foregoing methods are briefly outlined below.

2. The Method of Normal Modes*

Let the position of a mass element δ_m of the boat in the reference steady state be given by x_o, y_o, z_o and in its perturbed position at time t by x, y, z measured in the stability

---

*This treatment follows that of Reference 14.

axis system with origin at the mass center. Then the elastic displacements of the elements $x-x_o$, $y-y_o$, $z-z_o$ may be described by the following equations

$$
\begin{aligned}
x-x_o &= \sum_1^\infty f_n\,(x_o,\ y_o,\ z_o)\ \epsilon_n\,(t) \\
y-y_o &= \sum_1^\infty g_n\,(x_o,\ y_o,\ z_o)\ \epsilon_n\,(t) \\
z-z_o &= \sum_1^\infty h_n\,(x_o,\ y_o,\ z_o)\ \epsilon_n\,(t)
\end{aligned}
\qquad [64]
$$

where f_n, g_n, z_n give the shape of the n-th normal mode and ϵ_n is the generalized coordinate which gives the displacement of that mode. An important property of normal modes is that in each normal mode of free vibration the resultant angular and linear momenta are zero. Hence, the resulting elastic vibrations described by the superposition of normal modes produces no inertial coupling to the Euler equations of motion. Nevertheless the deformations of the boat will produce additional perturbation hydrodynamic forces which must be added to those due to the rigid body motions. Thus the linearized hydrodynamic perturbation Z-force, for example, would become, on summing over n

$$Z = Z_u u + Z_w w + ---- + \Sigma\, Z_{\epsilon n}\, \epsilon_n + \Sigma\, Z_{\dot{\epsilon} n}\, \dot{\epsilon}_n + \Sigma\, Z_{\ddot{\epsilon} n}\, \ddot{\epsilon}_n \qquad [65]$$

Similar additions would be made to the other force and moment equations.

The additional equations of equilibrium in the elastic degrees of freedom in terms of the generalized coordinates ϵ_n are obtained with the aid of Lagrange's equations of motion in a moving

frame of reference (see Reference 1)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\epsilon}_n} - \frac{\partial T}{\partial \epsilon_n} + \frac{\partial U}{\partial \epsilon_n} = F_n \qquad [66]$$

where T is the kinetic energy of the system relative to the moving frame of reference

U is the strain energy

$$F_n = \frac{\partial W}{\partial \epsilon_n}$$

W = work done on the system by the external forces (including the inertial forces, due to the motion of the frame of reference, which act on the system).

Since there is no inertial or elastic coupling between normal modes the above equations are coupled only through the hydrodynamic contributions to the F's. Thus the left hand side of Equation [66] for each normal mode is independent of any of the other normal modes. The kinetic energy of the n-th normal mode is therefore

$$T = \tfrac{1}{2} I_n \dot{\epsilon}_n^{\,2} \qquad [67]$$

where $I_n = \int (f_n^{\,2} + g_n^{\,2} + h_n^{\,2})\, dm$, the integral over the entire boat, is the generalized inertia in the n-th mode. For a system vibrating in an undamped natural mode the kinetic energy is a maximum and the strain energy is zero when the deformation is zero. When all elements are at the extreme position the strain energy is a maximum and the kinetic energy is zero. In this position the strain energy is equal to the maximum kinetic energy. Thus if $\epsilon_n = a \sin \omega_n t$, where ω_n is the natural frequency of the n-th mode, then

$$T_{max} = \tfrac{1}{2} I_n \omega_n^{\,2} a^2 = U_{max} \qquad [68]$$

Since the stress-strain relation is assumed to be linear $U = \frac{1}{2} k \epsilon_n^2$, where k is a constant, and $U_{max} = \frac{1}{2} k a^2$. Combining with Equation [68] $k = I_n \omega_n^2$ and

$$U = \frac{1}{2} I_n \omega_n^2 \epsilon_n^2 \qquad [69]$$

Inserting Equations [67] and [69] into Equation [66] gives

$$I_n \ddot{\epsilon}_n + I_n \omega_n^2 \epsilon_n = F_n \qquad [70]$$

In order to determine F_n it is necessary to evaluate the work done on the system, during a virtual displacement $\delta\epsilon_n$ by all the external forces acting on the system. Since there is no inertial coupling between the rigid body motions and the normal elastic modes, the work done on each of these modes by the inertial forces associated with non-uniform motion of the frame of reference is zero. The remaining contribution to the work done is that due to the hydrodynamic forces. Let the local normal-pressure perturbation on an element dS of the foil-strut system be $p(x_o, y_o, z_o)$ and let the local outward unit normal vector be $\vec{n}\ (n_x, n_y, n_z)$. Then the work done by the hydrodynamic forces in a virtual displacement is

$$\delta W_h = - \int p \vec{n} \cdot (\vec{r} - \vec{r}_o)\, dS \qquad [71]$$

where the integral is over the whole strut-foil system, and $\vec{r} - \vec{r}_o$ is the vector displacement at dS.

$$\vec{r} - \vec{r}_o = \Sigma\, (\vec{i}\, f_n + \vec{j}\, g_n + \vec{k}\, h_n)\, \delta\, \epsilon_n \qquad [72]$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors in the x, y, z directions.

Combining Equations [71] and [72] gives for F_n

$$F_n = \frac{\partial W_h}{\partial \epsilon_n} = - \int p \, (n_x f_n + n_y g_n + n_z h_n) \, dS \qquad [73a]$$

Now p is a function of all the generalized coordinates (including the rigid body perturbations). The result is that F_n is a linear function of all these variables which may be expressed in terms of a set of generalized hydrodynamic derivatives (or alternatively hydrodynamic transfer functions).

$$F_n = \frac{\partial F_n}{\partial u} u + \frac{\partial F_n}{\partial \dot{u}} \dot{u} + \text{---} + \frac{\partial F_n}{\partial p} p + \text{---} \frac{\partial F_n}{\partial \zeta} \zeta +$$

$$\text{---} + \sum_{m=1}^{\infty} \frac{\partial F_n}{\partial \epsilon_m} \epsilon_m + \sum_{m=1}^{\infty} \frac{\partial F_n}{\partial \dot{\epsilon}_m} \dot{\epsilon}_m + \sum_{m=1}^{\infty} \frac{\partial F_n}{\partial \ddot{\epsilon}_m} \ddot{\epsilon}_m \qquad [73b]$$

In practice, only the important derivatives would be retained. Some examples of these derivatives are given in Reference 3. The equations may be non-dimensionalized in the same manner as the rigid body equations.

3. Method of Assumed Modes

When the elastic deformation is described by a sum of artificially selected deformation shapes or modes, the resulting equations of motion exhibit inertial and elastic coupling between the modes. In order to simplify the illustration given below we assume only the vertical deflections are important. Thus the vertical displacement at a point x,y of the structure may be written

$$w(x,y,t) = \sum_{i=1}^{n} h_i \, (x,y) \, \epsilon_i(t)$$

where h_i describes the selected mode shapes and may also include the rigid body modes.

The kinetic energy of the system is

$$T = \tfrac{1}{2} \int \dot{w}^2 \, dm = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \, \dot{\epsilon}_i \, \dot{\epsilon}_j$$

where the integration is over all mass elements dm of the structure and

$$m_{ij} = \int h_i \, h_j \, dm$$

The strain energy of the system is expressed in terms of the stiffness influence coefficient $k\ (x,y,\xi,\eta)$, (see Reference 4, page 18), as

$$U = \tfrac{1}{2} \int w(x,y,t) \int k \, w(\xi,\eta,t) \, d\xi d\eta \, dx \, dy = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} \, \epsilon_i \epsilon_j$$

where the stiffness coefficient

$$k_{ij} = \int h_i(x,y) \int k \, h_j(\xi,\eta) d\xi d\eta dx \, dy$$

The stiffness coefficient associated with the rigid body modes vanish. Inserting T and U in Lagrange's equations gives

$$\sum_{j=1}^{n} m_{ij} \, \ddot{\epsilon}_j + \sum_{j=1}^{n} k_{ij} \, \epsilon_j = Q_i \qquad (i=1,2 \text{ ---} n) \qquad [74]$$

where $Q_i = - \int p \, n_z \, h_i \, dx \, dy$

In contrast to Equation [70] it is seen that due to the non-orthogonality of the assumed modes the inertial and elastic coupling terms m_{ij} and k_{ij} $(i \neq j)$ appear. In spite of this added complication it has often been shown that where the lifting surfaces have rigid chordwise sections it is more economical to choose artificial

modes because of the greater ease of computing Q_i (see pages 587-9 of Reference 4).

4. The Collocation Approach Using Matrices

Longitudinal Equations of Motion - The treatment given here is to illustrate briefly the general nature of the method. The reader is referred to Reference 5 for further details. It is assumed here that, for the longitudinal motion, only the transverse (z) deflections are of hydrodynamic importance. In the case where the structure is divided into a number of rigid segments, the torsion is, of course, of hydrodynamic importance also, but in the following formulation torsion is assumed to be taken into account by appropriate choice of the transverse elastic deflection $f_i^{(e)}$ with respect to the elastic axis (or other nodal line for twist). Moreover, by virtue of the fact that changes in the forward velocity perturbations u occur very slowly compared with the motions of the elastic perturbations the hydroelastic modifications of the derivatives in the X-force equation may be calculated on a quasi-steady basis. Therefore, only the Z-force and M-moment equations will be considered in the following treatment. For equilibrium of total forces along the z-axis

$$\int [Z_H - \rho_B \ddot{f}_t] \, d\tau = 0 \qquad [75]$$

where the integration is over the entire boat and

Z_H = time and space dependent hydrodynamic Z-forces per unit volume

ρ_B = mass density distribution of boat

$\ddot{f}_t$ = z-component of perturbation acceleration from equilibrium flight position of any point on boat

$d\tau$ = a volume element of the boat.

For equilibrium of X-forces and M-moments respectively

$$\int [X_H - \rho_B \ddot{g}_t] \, d\tau = 0 \qquad [76]$$

$$\int \left\{ [Z_H - \rho_B \ddot{f}_t] \, x - [X_H - \rho_B \ddot{g}_t] \, z \right\} d\tau = 0 \qquad [77]$$

where x and z are the x and z coordinates of element $d\tau$.

We next consider the boat to be subdivided into n elements. The z-component of the acceleration at element i of the boat is given by

$$\ddot{f}_{t_i} = - [Uq - \dot{w} + \dot{q}\, x_i - \ddot{f}_i^{(e)}] \qquad [78]$$

where the first three terms on the right are due to rigid body motion and the fourth term is due to elastic deformation. Hence the Z-force at element i is

$$Z_i = Z_{H_i} + m_i \, [Uq - \dot{w} + \dot{q}\, x_i - \ddot{f}_i^{(e)}] \qquad [79]$$

where Z_{H_i} = total hydrodynamic force on element i

m_i = mass of element i.

By considering the hydrofoil boat to be made up of n elements, Equation [75] may be rewritten as

$$\sum_{i=0}^{n} Z_i = \sum_{i=0}^{n} Z_{H_i} + m_i \, [Uq - \dot{w} + \dot{q}\, x_i - \ddot{f}_i^{(e)}] = 0 \qquad [80]$$

The difference between this equation and that for the rigid body is essentially due to the addition of the elastic deformation term $\ddot{f}_i^{(e)}$ both explicitly and in so far as Z_{H_i} depends on $f_i^{(e)}$ and its

spatial and time derivatives. The elastic deflection $f_i^{(e)}$ may be written in terms of the boat influence coefficients a_{ij} as

$$f_i^{(e)} = \sum_{j=0}^{n} a_{ij} Z_j \tag{81}$$

where a_{ij}, the free body influence coefficients of the boat, represent the elastic deflection at point i due to a unit force at point j.

Equation [81] represents a set of n linear differential equations. Written in matrix form these become

$$\{f_i^{(e)}\} = [a_{ij}] \{Z_j\} \tag{82}$$

where $\{f_i^{(e)}\}$ and $\{Z_j\}$ are the column matrices of the elastic deflection distribution and total force distribution (including inertial effects) given by Equation [79].

Methods for determining the influence coefficients are given in References 4, 5, and 6.

The local hydrodynamic Z-force column matrix may be conveniently broken up as follows

$$\{Z_{H_i}\}=\{Z_{R_i}\}-(\underline{A_{ij}} \frac{\partial^2}{\partial t^2} + \underline{B_{ij}} \frac{\partial}{\partial t} + \underline{C_{ij}})\{f_j^{(e)}\} + \{Z_{E_i}\} \tag{83}$$

where $\{Z_{R_i}\}$ represents the contribution of the rigid body motion on the hydrodynamic Z-force perturbation, determined by use of rigid body derivatives and/or with the aid of the square matrix $\underline{P}$.

$\underline{A_{ij}}$ = hydrodynamic added mass square matrix (this matrix may more conveniently be included in $\underline{P}$ if desired).

$$\underline{B_{ij}} = U^{-1}\ \underline{P}$$

$$\underline{C_{ij}} = \underline{P}\ \underline{D}$$

$\underline{P}$ = hydrodynamic square matrix by which the angle of attack column matrix $\{\alpha^{(e)}\}$ must be multiplied to obtain the corresponding column Z-force matrix. $\underline{P}$ may contain the differential time operator $\partial/\partial t$.

$\underline{D}$ is a streamwise differentiation square matrix defined in such a way that

$$\underline{D}\ \underline{()} = \{ \frac{\partial()i}{\partial x} \}$$

$$\{\alpha^{(e)}\} = (\frac{1}{U}\frac{\partial}{\partial t} + \underline{D})\ \{f^{(e)}\}$$

$\{Z_{E_i}\}$ = hydroelastic effect on hydrodynamic forces due to control surface input and seaway. These sources of Z-force will in general depend on the elastic deflections as well as rigid body motions. (Even when the control system is assumed known, the sensing element of an autopilot may be located on a flexible member.)

The determination of the terms $\underline{A}$, $\underline{B}$ and $\underline{C}$ (the subscript ij has been dropped for convenience), which represent the effect of varying local hull and foil distortion, velocity and acceleration in the elastic degrees of freedom, in general would involve the solution of integral equations. However, for approximate analysis, the use of strip theory can lead to useful results (see References 4-6).

Combining Equations [79], [82] and [83] yields

$$\{f^{(e)}\}=[a_{ij}] \left\{\{Z_{R_j}\}+\underline{m}\ \{Uq-\dot{w}+\dot{q}x_i\}-[\underline{C}+\underline{B}\frac{\partial}{\partial t} + (\underline{m}+\underline{A})\frac{\partial^2}{\partial t^2}]\{f^{(e)}\}+\{Z_{E_j}\}\right\} \quad [84]$$

where $\underline{m}$ is the diagonal mass distribution matrix.

Equations [84] represent a set of n equations which describe the motion of the boat in its elastic degrees of freedom. Equation [80] represents the Z-force equilibrium equation of the entire boat including the contribution of the elastic degrees of freedom. The force contribution, from the elastic degrees of freedom, $Z^{(e)}$, to the total Z-force is from Equations [80] and [83]

$$Z^{(e)} = - \underline{1}' \left\{ [(\underline{m}+\underline{A}) D^2 + \underline{B} D+\underline{C}] \{f^{(e)}\} + \{Z_E\} \right\} \qquad [85]$$

where $\underline{1}'$ is a unit row matrix which operates so as to sum all the terms of the column matrix on its right.

Since the hydroelastic forces in the x-direction were taken as quasistatic they will already be included in the rigid body equations of motion and since it is assumed that $g_i^{(e)}$ is negligible compared to $f_i^{(e)}$ the contribution of the elastic degrees of freedom to the rigid body moment are obtained by multiplying the terms in $f_i^{(e)}$ by $-x_i$, the longitudinal location of the center of pressure of the elastic element and summing. Thus

$$M^{(e)} = \underline{x}' \left\{ [(\underline{m}+\underline{A}) D^2 + \underline{B} D + \underline{C}] \{f^{(e)}\} + \{Z_E\} \right\} \qquad [86]$$

where $\underline{x}'$ is a row matrix of x.

When Equation [85 is added to the rigid body Z-force equation (Equation [51-b]) and Equation [86] is added to the rigid body M-moment equation ([Equation [51-c]), the resulting equations, together with Equation [51-a] and Equations [84], constitute a set of ordinary linear differential equations for the longitudinal motion of the elastic hydrofoil boat.

Lateral Equations of Motion - The elastic deformations of interest in connection with the lateral stability are of three types:

(a) The static, symmetric deflections in the z-direction which affect the equilibrium values of the angle of attack and dihedral distributions of the hydrofoils.

(b) The dynamic lateral deflections in the y-direction which affect all of the lateral stability derivatives.

(c) The dynamic, anti-symmetric deflections in the z-direction which are produced by the lateral motions but essentially affect only the rolling moment.

The deflections of type (a) may be taken into account in the rigid body equations directly and need not be considered further here. The deflections of type (b) and (c) will be treated separately below for convenience. The deflections of type (b) will be considered first.

The lateral force matrix may be written in an analogous manner to Equations [79] and [83]

$$\{Y_l\} = \{Y_{R_l}\} - \underline{m}\,\{\dot{v} + Ur - \dot{p}\, z_i + \dot{r}\, x_i - g\phi\}$$

$$- \left[\underline{C_l} + \underline{B_l}\,\frac{\partial}{\partial t} + (\underline{m}+\underline{A_l})\,\frac{\partial^2}{\partial t^2}\right] \{f_l^{(e)}\} + \{Y_{E_l}\} \qquad [87]$$

where $\underline{C_l}$, $\underline{B_l}$, $\underline{A_l}$ etc. have corresponding meaning to those in the longitudinal motion case but in general have different values. Also $\{f_l^{(e)}\}$ represents the lateral elastic deflection column matrix.

We let $[a_l]$ denote the structural square influence matrix for lateral elastic deflections so that

$$\{f_l^{(e)}\} = [a_l]\,\{Y_l\} \qquad [88]$$

Combining [87] and [88] gives the equations of motion for the lateral elastic deflections.

The contribution of the effect of the lateral elastic deflections to the Y-force, K-moment and N-moment rigid body equations are obtained from Equation [87] as

$$Y_1^{(e)} = -\underline{1}' \left\{ [(\underline{m}+\underline{A_1})D^2 + \underline{B_1}D + \underline{C_1}]\{f_1^{(e)}\} + \{Y_{E_1}\} \right\} \quad [89]$$

$$K_1^{(e)} = \underline{z_1}' \left\{ [(\underline{m}+\underline{A_1})D^2 + \underline{B_1}D+\underline{C_1}]\{f_1^{(e)}\} + \{Y_{E_1}\} \right\} \quad [90]$$

$$N_1^{(e)} = -\underline{x_1}' \left\{ [(\underline{m}+\underline{A_1})D^2 + \underline{B_1}D + \underline{C_1}]\{f_1^{(e)}\} + \{Y_{E_1}\} \right\} \quad [91]$$

where $\underline{z_1}'$ and $\underline{x_1}'$ are row matrices of z and x respectively.

For the antisymmetric elastic effects the only contribution from the rigid body motion are the pure roll terms. The force distribution is given by the column matrix

$$\{Z_2\}=\{Z_{R_2}\}-\underline{m}\{\dot{p}y_i\}-[\underline{C_2}+\underline{B_2}D + (\underline{m} + \underline{A_2})D^2]\{f_2^{(e)}\} +\{Z_{E_2}\} \quad [92]$$

where $\{Z_{R_2}\}$ represents the antisymmetric vertical component of the rigid body hydrodynamic force distribution

$\underline{C_2}$, $\underline{B_2}$, $\underline{A_2}$ are the antisymmetric hydrodynamic influence matrices

$\{f_2^{(e)}\}$ is the z-component of the elastic deflection.

We let $[a_2]$ denote the structural influence matrix for the antisymmetric elastic deflections so that

$$\{f_2^{(e)}\} = [a_2]\{Z_2\} \quad [93]$$

Combining Equations [92] and [93] gives the equations of motion.

The total rolling moment contributed by $f_2^{(e)}$ to the rigid body K-moment equation is

$$K_2^{(e)} = -\underline{y_2'}\left\{[(\underline{m}+\underline{A_2})D^2+\underline{B_2}D+\underline{C_2}]\{f_2^{(e)}\}+\{Z_{E_2}\}\right\} \qquad [94]$$

The contributions of the elastic degrees of freedom which must be added to the rigid body equations are finally $Y_1^{(e)}$, $K_1^{(e)}+K_2^{(e)}$ and $N_1^{(e)}$ respectively.

Quasistatic Equations - When $\underline{B}\ \frac{\partial}{\partial t}$ and $(\underline{m}+\underline{A})\ \frac{\partial^2}{\partial t^2}$ are sufficiently small as they usually will be when the motions of the boat as a whole are of primary interest (this may not be true for high speed motions in head seas) then it is permissible to drop these terms in Equations [84]. This amounts to saying that the elastic deflections are in phase with the loads that produce them. Solving for $\{f^{(e)}\}$

$$\{f^{(e)}\}=(\underline{I}+[a_{ij}]\ \underline{C})^{-1}[a_{ij}]\left\{\{Z_{R_j}\}+\underline{m}\ \{Uq-\dot{w}+\dot{q}\ x_i\}+\{Z_{E_j}\}\right\} \qquad [95]$$

where $\underline{I}$ denotes the unit diagonal matrix.

Now from Equation [85] we have, on combining with Equation [95]

$$Z^{(e)} = -\underline{1}'\ \underline{C}\ (\underline{I}+[a_{ij}]\underline{C})^{-1}[a_{ij}]\left\{\{Z_{R_j}\}+\underline{m}\ \{Uq-\dot{w}+\dot{q}\ x_i\}+\{Z_{E_j}\}\right\} - \underline{1}'\ \{Z_{E_j}\} \qquad [96]$$

On adding the above value for $Z^{(e)}$ to the right hand side of the rigid body Z-force equation we obtain the modified rigid body Z-force equation in which quasi-static hydroelastic effects are included. An analogous procedure is followed in modifying the other rigid body equations.

Steady Flight Hydroelastic Effects - The steady flight hydroelastic problem is a special case of the above problem. Neglecting the rigid body perturbation terms in u, v, w, etc. and denoting the static hydrodynamic rigid body Z-force distribution by $\{Z^{(r)}\}$ and the weight distribution by $\{W\}$ then the Z-force distribution modified by the elastic deformation of the foils and boat is

$$\{Z\} = - \underline{C}\ \{f^{(e)}\} + \{Z^{(r)}\}$$

$$= - \underline{C}\ (I+[a_{ij}]\ \underline{C})^{-1}\ [a_{ij}](\{Z^{(r)}\} - \{W\}) + \{Z^{(r)}\} \qquad [97]$$

Similarly static control effectiveness may be calculated. If we put $\{Z_\delta^{(r)}\}$ as the control rate distribution for the rigid control then the control rate distribution modified by the elastic deformation (neglecting deformations due to weight distribution) is

$$\{Z_\delta\} = - \underline{C}\ \{f^{(e)}\} + \{Z_\delta^{(r)}\}$$

$$= - \underline{C}\ (I + [a_{ij}]\ \underline{C})^{-1}\ [a_{ij}]\{Z_\delta^{(r)}\} + \{Z_\delta^{(r)}\} \qquad [98]$$

5. Applications

An attempt has been made in the foregoing sections to present the essential features of the three principle methods of hydroelastic analysis, the normal mode, the assumed mode, and the collocation methods. The treatment has been necessarily brief and sketchy and the reader is referred to the standard references for fuller details and examples of their application. In the first two methods the deformation is assumed as a super-position

of a finite number of normal or assumed modes of the structure and in the third the deformation is described by the deflections at a number of discrete points on the structure. Most of the methods used in the past have been based on the modal description. However, for very complex configurations, and with the increasing speed and capacity of digital computers the collocation approach using matrices may prove more feasible.

Static Stability and Control - Quite obviously static stability derivatives such as Z_w, M_w, etc. can be affected by the variation in the elastic deflection of hydrofoils and struts with load. The importance of the possibilities of divergence of struts and hydrofoils cannot be overemphasized. The hydroelastic effects on control effectiveness and reversal is, of course, also an important design consideration. Examples of such computations appear in many references (see References 4-8).

Dynamic Stability and Control - As stated earlier, if the first natural bending frequency of the strut or hydrofoil (taking into account the added mass of the water) is relatively large compared to those of the rigid body modes, the deflections of the elastic structure are essentially in phase with the perturbation loads and the rigid body stability derivatives need be corrected only for the static elastic deflection produced by the loads. If the first bending frequency is approximately equal to that of one of the rigid body modes then the correction to the rigid body stability derivatives should include the dynamic hydroelastic effects. Many examples of the application of aeroelastic methods to such problems in the case of aircraft are given

in the aeronautical literature (see Reference 5).

Response to External Forces - When the dynamic response of a hydrofoil boat is a result of rapidly applied forces such as those occurring in a rough sea, or the abrupt deflection of control surfaces, the boat is caused not only to translate and rotate but structural vibrations may occur. The latter may have a significant effect on the internal stress distribution in the structure and on the over-all response of the boat. This problem has received wide attention in the aircraft field in relation to loading and control problems in rough weather, gust alleviation, landing loads, etc. Examples of applications of aeroelastic analysis to such problems are contained in many references (see References 4-7).

Flutter - Flutter is a dynamic instability which is brought about by the hydroelastic properties of the craft. It may involve the unrestrained motions of the entire craft. This is likely to be the case when the first bending frequency is approximately equal to one of the rigid body modes. It is then necessary to include the rigid body modes in the flutter determination. When the inertia of the strut or hydrofoil is of the same order of magnitude as the hull, the rigid body modes should be included. For example it may be necessary to include the rigid body roll degree of freedom in an analysis of strut flutter. It is perhaps much more likely, however, that particular modes will concentrate in the struts or hydrofoils. When this occurs one can study the stability of such modes without consideration of the rigid body modes. One may, for example, regard the hydrofoil

support as stationary and rigid.

It was long considered that flutter of subcavitating hydrofoils was extremely unlikely because of the low structural-to-fluid density ratio (9), (10). More recently, however, investigators (8), (11), (12), (15), and (16) have demonstrated that the flutter of subcavitating struts and hydrofoils can be a serious problem. In addition Kaplan and Henry (13) have shown that the same may also be true for supercavitating hydrofoils.

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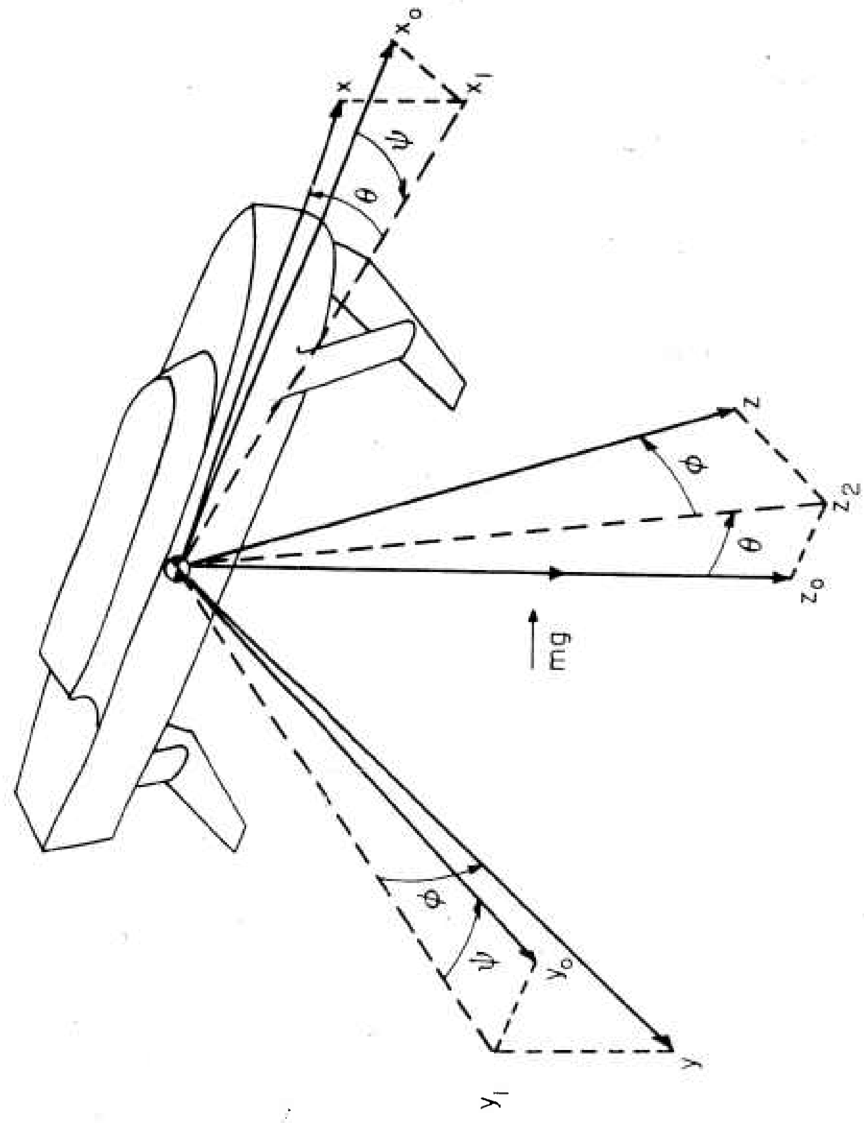

Figure 1- Relation Between Body Axes and Fixed Axes

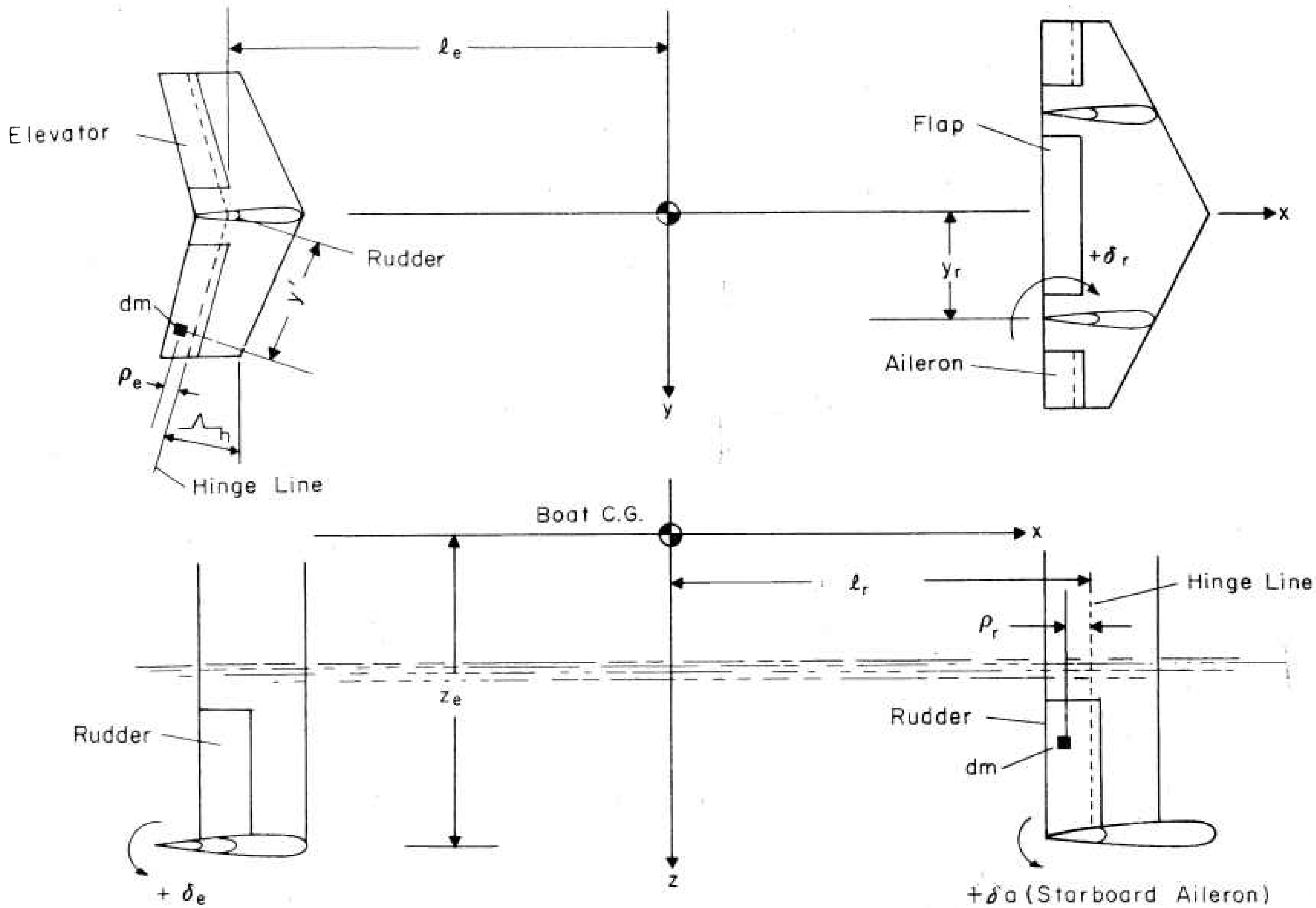

Figure 2– Schematic of Elevator, Rudder, Aileron, Flap Arrangement

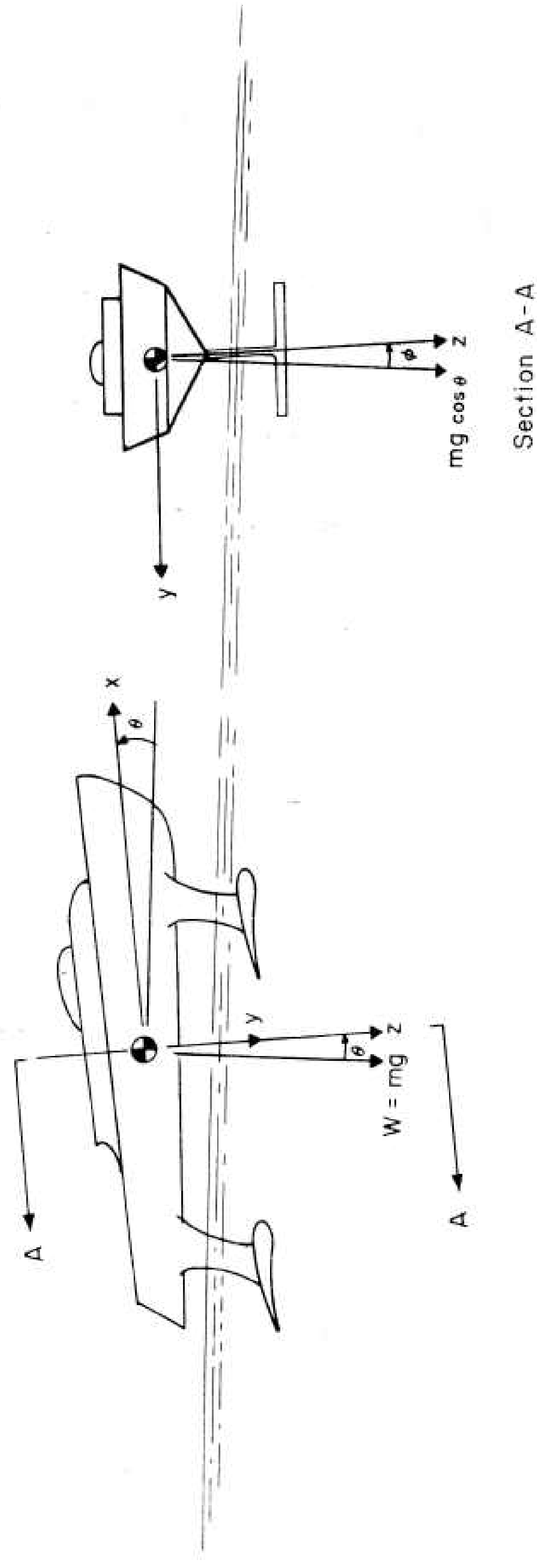

Figure 3- Gravity Force Components

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